

GENERALIZED GROUP PRESENTATION AND FORMAL DEFORMATIONS OF CW COMPLEXES

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ABSTRACT. A Peiffer-Whitehead word system \mathscr{W} , or generalized group presentation, consists of generators, relators (words of order 2), and words of higher order n that represent elements of a free crossed module ($n = 3$) or a free module ($n > 3$). The P_n -equivalence relation on word systems generalizes the extended Nielsen equivalence relation on ordinary group presentations. Word systems, called homotopy readings, can be associated with any connected CW complex K by removing a maximal tree and selecting one generator or word per cell, via relative homotopy. Given homotopy readings \mathscr{W}_1 and \mathscr{W}_2 of finite CW complexes K_1 and K_2 respectively, we show that \mathscr{W}_1 is P_n -equivalent to \mathscr{W}_2 if and only if K_1 formally $(n + 1)$ -deforms to K_2 . This extends results of P. Wright (1975) and W. Metzler (1982) for the case $n = 2$. For $n \geq 3$, it follows that \mathscr{W}_1 is P_n -equivalent to \mathscr{W}_2 if and only if K_1 and K_2 have the same simple homotopy type.

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1.1 Introduction. The purpose of this paper is to reintroduce J. H. C. Whitehead's homotopy systems from the viewpoint of R. Peiffer. In [8], extended group presentations ("Σ-systems") are considered that have three fields, namely generators, relators and identities among relators ("relation identities"). At the same time, Whitehead [14] introduced the homotopy system of a CW complex K , consisting of the fundamental group $\rho_1 = \pi_1(K^1)$ and the relative homotopy groups $\rho_n = \pi_n(K^n, K^{n-1})$ ($n > 1$) with the standard boundary maps $d_n: \rho_n \rightarrow \rho_{n-1}$. We generalize Peiffer's approach using homotopy systems as a guide, by defining generalized group presentations, or *word systems*, with an arbitrary number of component fields. Each entry or *word* in such a field gives rise to an element in the free module ρ_n , much as a relator represents an element of a free group in an ordinary group presentation.

After setting up some machinery, we prove the following generalization of P. Wright's Theorem [15].

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Theorem (1.1.1). (See (6.1.1).) *Two CW complexes K, L of dimension at most n formally $(n+1)$ -deform to each other, $K \xrightarrow{n+1} L$, iff any corresponding word systems $\mathscr{W}(K)$ and $\mathscr{W}(L)$ are P_n -equivalent, as defined in Subsection 2.3.*

Here, $K \xrightarrow{n+1} L$ means that there is a sequence of elementary expansions and collapses of dimension at most $n+1$ that transforms K to L , and P_n -equivalence is a generalization of the equivalence relation introduced in [1].

The proof of this theorem in the more difficult case ($\xrightarrow{n+1} \Rightarrow P_n$) ultimately rests on an analysis of the structure of words in a word system. In fact, if e is an n -cell in an intermediate complex K' that is collapsed during the deformation $K' \xrightarrow{n+1} L$, then the “spelling” of the word corresponding to e in $\mathscr{W}(K')$ may be used as a prescription for converting the collapse of e into a P_n -equivalence of word systems. See Theorem (4.1.2).

In view of [12], we may conclude that simple homotopy equivalence of CW complexes of dimension $n \geq 3$ is characterized by P_n -equivalence of corresponding word systems. For $n = 2$, this statement would be equivalent to the Andrews-Curtis conjecture [1]. It is possible that an analysis of P_3 -equivalence of word systems of order 2 would shed some light on that conjecture.

Word systems offer a flexible means to express geometric deformations in an algebraic manner. For example, in order to manage the effect of geometric 3-deformations on corresponding group presentations, Wright transformed a given 3-deformation of 2-complexes into a transient deformation in which at most one 3-cell is present at any time. In contrast, the proof of Theorem (1.1.1) permits an accumulation of $(n+1)$ -cells in an $(n+1)$ -deformation: the words in a word system express the information needed to carry out the proof.

Theorem (1.1.1) can be used to show that Wright’s transiency property holds in higher dimensions. See Proposition (7.1.2).

This work requires a technical result that may be of interest, concerning 3-dimensional cells in a CW complex. If e_1^2, \dots, e_n^2 are free faces of a 3-cell e^3 , then there exists a characteristic map ϕ of e^3 that “realizes the freeness” of all of those faces simultaneously, i.e., ϕ restricts to a characteristic map for each 2-cell e_i^2 (see Proposition (3.1.2)). This fills an apparent gap in S. Young’s proof of Wright’s Theorem for the CW category [16]. Our proof requires a theorem of Dyer and Hamstrom [6].

This paper is organized as follows. The remainder of §1 reviews preliminary material. We define the notion of a word system or generalized group presentation in §2. The correspondence between CW complexes and word systems is discussed. Various equivalence relations patterned after EN- and Q-moves (see Subsection 1.3) are proposed for word systems. In §3, several technical results are established. In §§4 and 5, certain of the equivalence relations defined in §2 are shown to have identical equivalence classes, which proves most of Theorem (1.1.1). The proof of that theorem is completed in §6. Section 7 contains applications.

I am indebted to Professor Craggs, who directed this dissertation and suggested the problem of proving Wright’s Theorem using Σ -systems. I am grateful to the referee for his comments, which were helpful, and for suggesting the transiency problem addressed by Proposition (7.1.2).

1.2 Notation and conventions. For elements a and b in a topological space X , a *path* $p: a \rightsquigarrow b$ from a to b is a continuous function $p: I \rightarrow X$ that satisfies $p(0) = a$, $p(1) = b$.

We are interested in finite, connected CW complexes K . The m -skeleton K^m of K is the subcomplex consisting of all cells in K of dimension at most m . Let B^m denote a closed topological m -ball, and suppose that μ is the index of an m -cell in K . Unless other indicated, e^m will always be understood to be an m -cell in a CW complex, $\phi^m: B^m \rightarrow K$ will be a characteristic map for $e^m = \phi^m(\text{int } B)$, and f^m will be the attaching map $f^m = \partial\phi^m: \text{bdy } B^m \rightarrow K^{m-1}$. The same relationships will hold for decorated symbols, e.g., $\bar{e}_\mu^m, \bar{\phi}_\mu^m, \bar{f}_\mu^m$.

Proposition (1.2.1). Suppose that $\phi, \psi: B^m \rightarrow K^m$ are two characteristic maps for the same m -cell $e^m \in K$ of a CW complex K that agree on the basepoint $*$ of B^m , $\phi(*) = \psi(*) = *_K$.

Then the relative homotopy classes $[\phi]$ and $[\psi]$ in $\pi_m(K^m, K^{m-1}, *_K)$ agree up to orientation reversal and the action of $\pi_1(K^{m-1}, *_K)$, that is,

$$[\phi] = \alpha \cdot [\psi]^\varepsilon \quad \text{for some } \alpha \in \pi_1(K^{m-1}, *_K), \quad \varepsilon = \pm 1.$$

Proof. This is easily proved using [13, V(1.4)] and the Relative Hurewicz Theorem. \square

Given two maps $f_0, f_1: (X, A, *_X) \rightarrow (Y, B, B)$ and a path $p: f_0(*_X) \rightsquigarrow f_1(*_X)$ in B , we say that f_0 is *freely homotopic* to f_1 via the path p , $f_0 \simeq_p f_1$, if there is a homotopy $H: X \times I \rightarrow Y$ satisfying $H|_{X \times 0} = f_0$, $H|_{X \times 1} = f_1$, $H|_{A \times I} \subset B$, and $H|_{*_X \times I} = p$. The symbol \simeq denotes based homotopy, i.e., free homotopy via the constant path. The notation $[f]$ will generally be used to denote the (absolute or relative) based homotopy class of a map.

A *topological ball tuple* $(B^m, B_1^{m-1}, \dots, B_Q^{m-1})$ is a closed m -ball together with Q disjoint closed $(m-1)$ -balls $B_1^{m-1}, \dots, B_Q^{m-1}$ that are contained in $\text{bdy}(B^m)$. Suppose that $e^m, e_1^{m-1}, \dots, e_Q^{m-1}$ are cells in a CW complex K . We say that a characteristic map $\phi^m: B^m \rightarrow K$ is a *multiple collapse map* for $(e^m, e_1^{m-1}, \dots, e_Q^{m-1})$ if there is a ball tuple structure $(B^m, B_1^{m-1}, \dots, B_Q^{m-1})$ on B^m such that

- $\phi^m|_{B_q^{m-1}}$ is a characteristic map for the $(m-1)$ -cell e_q^{m-1} , and
- $\phi^m|(B^m - B_q^{m-1})$ does not meet e_q^{m-1} .

In the case $Q = 1$, we call ϕ^m a (single) *collapse map*.

Let K and L be CW complexes satisfying $L = K \cup e^{m-1} \cup e^m$, and suppose that a collapse map ϕ^m exists for (e^m, e^{m-1}) . The move $[K, L]$ is an *elementary m -expansion*, written $K \nearrow_{e,m} L$. The move $[L, K]$ is an *elementary m -collapse*, $L \searrow_{e,m} K$. An *elementary m -deformation* is an elementary m -expansion or m -collapse. An elementary m -deformation is said to have *dimension m* .

A *formal m -deformation*, $K \xrightarrow{m} L$, is a finite sequence of complexes $K = K_0, K_1, \dots, K_Q = L$ such that each move $[K_q, K_{q+1}]$ is an elementary deformation of dimension at most m . That formal m -deformation is *transient* if K and L are $(m-1)$ -complexes and every m -cell is collapsed immediately after it is expanded. When no deformation $[K_q, K_{q+1}]$ has dimension less than n ,

we write $K \xrightarrow[n]{m} L$. If all those moves are elementary expansions or collapses, we write $K \nearrow^m L$ or $K \searrow^m L$.

Definition (1.2.2). A move $[K, L]$ is an *elementary homotopy m -move*, $K \xrightarrow[e, m]{} L$, if there are cells $e^m \in K$ and $\bar{e}^m \in L$ such that

- (a) $K = K' \cup e^m$ and $L = K' \cup \bar{e}^m$ for some common subcomplex K' , and
- (b) those cells have attaching maps f^m and \bar{f}^m that are freely homotopic in K' .

Such a homotopy move is said to have *dimension m* . If the image of the homotopy in condition (b) is contained in a subcomplex $K_0 \subset K'$, we say that the homotopy move *takes place within K_0* . By the Cellular Approximation Theorem, we may adjust any homotopy satisfying condition (b) to insure that the homotopy move takes place within the complex $(K_0)^m$.

A sequence $K = K_1, K_2, \dots, K_Q = L$ is a *homotopy m -deformation*, $K \xrightarrow{m} L$, if each move $[K_q, K_{q+1}]$ is an elementary expansion, collapse or homotopy move, of dimension at most m .

Suppose that $K \xrightarrow[e, m]{} L = K \cup \bar{e}^m - e^m$ is an elementary homotopy m -move that takes place within $K^m - e^m$. Then one may attach an $(m+1)$ -cell e^{m+1} to $K^m \cup \bar{e}^m$ via the attaching map

$$f^{m+1} = h \cup (\phi^m \times 0) \cup (\bar{\phi}^m \times 1): \text{bdy}(\mathbf{I}^{m+1}) \rightarrow K^m \cup \bar{e}^m$$

where $h: \text{bdy}(\mathbf{I}^m) \times \mathbf{I} \rightarrow K^m - e^m$ is the given free homotopy between the attaching maps. The $(m+1)$ -expansion $K \nearrow_{e, m+1} K \cup \bar{e}^m \cup e^{m+1}$ is called the

expansion associated with the homotopy move $[K, L]$. Likewise, $K \cup \bar{e}^m \cup e^{m+1} \searrow_{e, m+1} (K - e^m) \cup \bar{e}^m$ is called the *collapse associated with $[K, L]$* . We also

refer to that homotopy move as *sliding the m -cell e^m across the $(m+1)$ -cell e^{m+1} or sliding the attaching map f^m across the homotopy h* .

Lemma (1.2.3). Let K and L be CW n -complexes. If a homotopy n -deformation $K \xrightarrow{n} L$ exists, then there is a transient formal $(n+1)$ -deformation $K \xrightarrow{n+1} L$.

Proof. Replace each homotopy m -move $K_q \xrightarrow[e, m]{} K_{q+1}$ in the given homotopy n -deformation by the associated expansion and collapse

$$K_q \nearrow_{e, m+1} K \cup \bar{e}^m \cup e^{m+1} \searrow_{e, m+1} (K - e^m) \cup \bar{e}^m. \quad \square$$

1.3 Extended Nielson transformations and Wright's Theorem. We now briefly review group presentations and 2-dimensional complexes in order to motivate higher-dimensional definitions and relate our results to the literature.

A finite group presentation $\mathcal{P} = \langle y_1, \dots, y_J \mid R_1, \dots, R_K \rangle$ consists of generators y_1, \dots, y_J , which are distinct symbols, and relators R_1, \dots, R_K , which are words in the symbols y_j and their inverses. The group presented by \mathcal{P} is

$$G(\mathcal{P}) = F(y_1, \dots, y_J) / \text{n.cl.}(\{R_1, \dots, R_K\}).$$

Two words are *equivalent*, $w_1 \sim w_2$, if they represent the same element $\langle w_1 \rangle = \langle w_2 \rangle \in G(\mathcal{P})$. Two words are *congruent*, $w_1 \equiv w_2$, if they are identical as

sequences of symbols. The presentations $\mathcal{P} = \langle y_1, \dots, y_J \mid R_1, \dots, R_K \rangle$ and $\mathcal{P}' = \langle y'_1, \dots, y'_M \mid R'_1, \dots, R'_N \rangle$ are *congruent* if they have the same cardinalities $J = M$, $K = N$ and if each relator R'_k arises from the corresponding relator R_k by substitution, $R'_k \equiv R_k(y'_1, \dots, y'_J)$. Words or presentations are considered to be the same only if they are congruent.

There are two common approaches to defining extended Nielsen transformations on group presentations. The moves for each approach are listed below.

EN-moves [8, 15]. Let $\mathcal{P} = \langle y_1, \dots, y_M \mid R_1, \dots, R_N \rangle$ be a group presentation.

- EN1. *Change of representing word.* For some n , replace R_n by a word \bar{R}_n that represents the same element in the free group $F(y_1, \dots, y_M)$.
- EN2. *Inversion of a relator.* For some n , replace R_n by the inverted word $(R_n)^{-1}$.
- EN3. *Product of relators.* For some n and j ($n \neq j$), replace R_n by $R_n R_j$.
- EN4. *Interchange of relators.* For some n and j ($n \neq j$), replace R_n by R_j and R_j by R_n .
- EN5. *Conjugation of relators.* For some m and n , replace R_n by $y_m R_n y_m^{-1}$.
- EN6. *Expansion.* Append a new generator y_{M+1} and a new relator $R_{N+1} \equiv y_{M+1} R$ for some word $R = R(y_1, \dots, y_M)$ in the other generators.
- EN7. *Collapse.* Remove y_M and R_N . This move is permitted only if $R_N \equiv y_M R$ for a word $R = R(y_1, \dots, y_{M-1})$ in the other generators, and if y_M does not occur in R_1, \dots, R_{N-1} .

Q-moves [9, 7]. Let $\mathcal{P} = \langle y_1, \dots, y_M \mid R_1, \dots, R_N \rangle$ be a group presentation.

Q1 = EN1, Q2 = EN2, Q3 = EN3, Q4 = EN5.

- Q5. *Automorphism of underlying free group.* For an automorphism σ of the free group $F(y_1, \dots, y_M)$, replace the word $R_n(y_1, \dots, y_M)$ by the word $R_n(\sigma y_1, \dots, \sigma y_M)$ for all n .
- Q6. *Stabilization.* Append a new generator y_{M+1} and a new relator $R_{N+1} \equiv y_{M+1}$.
- Q7. *Destabilization.* Remove y_M and R_N . This move is permitted only if $R_N \equiv y_M$ and y_M does not occur in R_1, \dots, R_{N-1} .

Two group presentations are said to be *EN-equivalent* if one can be obtained from the other by a sequence of moves of the types EN1, ..., EN7. Two group presentations are *Q-equivalent* if they are related by a sequence of moves of types Q1, ..., Q4; they are *Q*-equivalent* if they are related by a sequence of Q1, ..., Q5 moves; and they are *Q**-equivalent* if they are related by a sequence of Q1, ..., Q7 moves.

Proposition (1.3.1) (see [4, Appendix and 7]). *Two group presentations are EN-equivalent if and only if they are Q**-equivalent.* \square

Suppose that K is a CW complex of dimension two. A group presentation is a *cellular presentation* for $\pi_1(K)$ if its generators correspond to a basis of the free group $\pi_1(K^1, *_K)$ and its relators are words representing the elements induced by attaching maps of the 2-cells in K . More precisely, each generator y_m is associated with the homotopy class $[\lambda_m] \in \pi_1(K^1, *_K)$ of a loop $\lambda_m: (I, \text{bdy}(I)) \rightarrow (K^1, *_K)$. Each relator $R_n = R_n(y_1, \dots, y_M)$ represents a homotopy class $[\xi_n \# f_n^2 \# \bar{\xi}_n] \in \pi_1(K^1, *_K)$, where $f_n^2: \text{bdy}(I^2) \rightarrow K^1$ is an

attaching map for the 2-cell $e_n^2, \xi_n: *_K \rightsquigarrow f_n^2(0)$ is a connecting path to the basepoint of e_n^2 , $\#$ denotes path multiplication and $\bar{\xi}_n$ denotes the inverse path $\bar{\xi}_n(t) = \xi_n(1-t)$ ($0 \leq t \leq 1$).

The Seifert-Van Kampen Theorem implies that a cellular presentation for K does in fact present the group $\pi_1(K, *_K)$. By a standard construction, every group presentation is a cellular presentation for some 2-complex $K(\mathcal{P})$.

Proposition (1.3.2). *Suppose that $\mathcal{P} = \langle y_1, \dots, y_M \mid R_1, \dots, R_N \rangle$ and $\mathcal{P}' = \langle y'_1, \dots, y'_M \mid R'_1, \dots, R'_N \rangle$ are cellular presentations for the same 2-complex K .*

Then \mathcal{P} and \mathcal{P}' are Q^ -equivalent.*

Proof. Straightforward. \square

The following is a consequence of our Theorem (6.1.1).

Theorem (1.3.3) (Wright's Theorem). *Suppose that K and K' are finite, connected CW 2-complexes. Let \mathcal{P} be a cellular presentation of K and \mathcal{P}' a cellular presentation of K' .*

Then K formally 3-deforms to K' , $K \overset{3}{\frown} K'$ if and only if \mathcal{P} is EN-equivalent to \mathcal{P}' .

Wright proved the result in the polyhedral category. Metzler and his students have established the CW case (unpublished; see [7, p. 293, Note (2)]). Both Wright and Metzler used transient deformation in their proofs of sufficiency, i.e., a given formal deformation must be modified so that every 3-cell is collapsed immediately after it is introduced.

2. GENERALIZED GROUP PRESENTATIONS

2.1 Word systems. Let G be a group. A G -crossed module is a group Λ together with an action \cdot of G on Λ and a homomorphism $d: \Lambda \rightarrow G$ such that

- (a) $\lambda_1 \lambda_2 \lambda_1^{-1} = d(\lambda_1) \cdot \lambda_2$ for $\lambda_i \in \Lambda$, and
- (b) $d(g \cdot \lambda) = g d(\lambda) g^{-1}$ for $g \in G, \lambda \in \Lambda$.

Crossed modules were invented by J. H. C. Whitehead [14] to express the structure of second relative homotopy groups of complexes. See [10] for a general exposition of the properties of crossed modules. We observe that an R -module M is always an R -crossed module with trivial homomorphism d . However, the groups G and Λ for a crossed module need not be abelian.

Maps $f = (f_\Lambda, f_G): (\Lambda, G, \cdot, d) \rightarrow (\Lambda', G', \cdot', d')$ between crossed modules are defined in the obvious way. A *free crossed module* is a free object in the category of crossed modules. Whitehead [14] gave the following construction of a free crossed module. Let $F(X)$ denote the free group on the set X . Choose elements g_α ($\alpha \in \mathcal{J}$) of G , and consider the subgroup P of $F(G \times \mathcal{J})$ defined by

$$P = \text{n.cl.}\{(g, \alpha)(h, \beta)(g, \alpha)^{-1}(g g_\alpha g^{-1} h, \beta)^{-1} \mid g, h \in G; \alpha, \beta \in \mathcal{J}\}.$$

Then the quotient $\Lambda(\mathcal{J}) = F(G \times \mathcal{J})/P$, together with the G -action induced by $g \cdot (g', \alpha) = (g g', \alpha)$ and the homomorphism $d: \Lambda(\mathcal{J}) \rightarrow G$ induced by $(g, \alpha) \rightarrow g g_\alpha g^{-1}$, forms a free G -crossed module. We call $\Lambda(\mathcal{J})$ the *standard free G -crossed module with basis $\{(1, \alpha) \mid \alpha \in \mathcal{J}\}$ and boundaries g_α*

$(\alpha \in \mathcal{J})$. Elements of the subgroup P are called *Peiffer identities*. Elements $\langle w \rangle \in \Lambda(\mathcal{J})$ are cosets of words of the form

$$(2.1.1) \quad w = (g_1, \alpha_1)^{\varepsilon_1} (g_2, \alpha_2)^{\varepsilon_2} \cdots (g_n, \alpha_n)^{\varepsilon_n}$$

where $\varepsilon_i = \pm 1$.

Let $\mathbf{Z}G$ be the integral group ring of G . Suppose $M(\mathcal{J})$ is a free G -module with basis $\{m_\alpha \mid \alpha \in \mathcal{J}\}$, and let (g, α) denote the element $g \cdot m_\alpha \in M(\mathcal{J})$. Then we may represent the elements $\langle w \rangle \in M(\mathcal{J})$ by words of the form

$$(2.1.2) \quad w = \varepsilon_1(g_1, \alpha_1) + \varepsilon_2(g_2, \alpha_2) + \cdots + \varepsilon_m(g_m, \alpha_m)$$

where $\varepsilon_i = \pm 1$, $g_i \in G$, $\alpha_i \in \mathcal{J}$.

A *syllable* is an element $(g_1, \alpha_1)^{\varepsilon_1}$ or $\varepsilon_i(g_i, \alpha_i)$ appearing in words of the form (2.1.1) or (2.1.2). For convenience, we sometimes use the additive form (2.1.2) rather than the multiplicative form (2.1.1) for words that represent elements of a crossed module, even though crossed modules are not commutative in general.

For an m -cell e^m in a pointed CW complex pair $(X, A, *)$, let ϕ^m be a characteristic map for e^m and let ξ^m be a path from $*$ to the basepoint of e^m . Theorems (2.1.3) and (2.1.4) below relate free crossed modules and free modules to relative homotopy groups.

Theorem (2.1.3). (See [14, §16] and [2].) *Suppose that X is obtained from a connected topological space A by adjoining 2-cells e_1^2, \dots, e_j^2 .*

Then the relative homotopy group $\pi_2(X, A)$ is a free $\pi_1(A)$ -crossed module, and there is an isomorphism of free $\pi_1(A)$ -crossed modules

$$\sigma_2: \pi_2(X, A) \cong \Lambda(\{1, \dots, J\})$$

defined by $\sigma_2([\xi_j^2 \# \phi_j^2 \# \bar{\xi}_j^2]) = \langle (1, j) \rangle$. \square

Theorem (2.1.4). (See [14, §16] and [13, V.1].) *Suppose that X is obtained from a connected topological space A by adjoining m -cells e_1^m, \dots, e_j^m for some dimension $m \geq 3$.*

Then the relative homotopy group $\pi_m(X, A)$ is a free $\mathbf{Z}\pi_1(A)$ -module, and there is an isomorphism of free $\mathbf{Z}\pi_1(A)$ -modules

$$\sigma_m: \pi_m(X, A) \cong M(\{1, \dots, J\})$$

defined by $\sigma_m([\xi_j^m] \cdot [\phi_j^m]) = \langle (1, j) \rangle$. \square

The isomorphisms of these theorems will be called *Whitehead isomorphisms*.

We wish to generalize the notion of a finite group presentation by permitting more than two fields. The entries in the new fields will be words w_σ^s ($s > 2$) representing elements in a free module or free crossed module. Such words may correspond to boundaries of higher dimensional cells e_σ^s of a CW complex, much as relators correspond to the boundaries of 2-cells in a cellular presentation.

Definition (2.1.5). A *Peiffer-Whitehead word system* or *extended group presentation* \mathcal{W} consists of a finite list of finite sets $\langle \mathcal{W}^{(1)} | \mathcal{W}^{(2)} | \dots | \mathcal{W}^{(n)} \rangle$ together with *boundary homomorphisms* d_3, \dots, d_n , described as follows.

$\mathcal{W}^{(1)} = \mathcal{J}_1$ is a set of indices, called 1-indices.

$\mathscr{W}^{(2)} = \{w_\beta^2 \mid \beta \in \mathcal{J}_2\}$ is a set of words representing elements of the free group $F = F(\mathcal{J}_1)$.

$\mathscr{W}^{(3)} = \{w_\gamma^3 \mid \gamma \in \mathcal{J}_3\}$ is a set of words representing elements of the standard free F -crossed module $\Lambda = \Lambda(\mathcal{J}_2)$ with boundaries $\{c_\beta = \langle w_\beta^2 \rangle \mid \beta \in \mathcal{J}_2\}$.

$\mathscr{W}^{(m)} = \{w_\mu^m \mid \mu \in \mathcal{J}_m\}$ ($4 \leq m \leq n$) is a set of words representing elements of the standard free $\mathbf{Z}G$ -module $M_m = M_m(\mathcal{J}_{m-1})$, where $G = F/\text{n.cl.}(\{\langle w_\beta^2 \rangle \mid \beta \in \mathcal{J}_2\})$ is the group presented by $\langle \mathscr{W}^{(1)} \mid \mathscr{W}^{(2)} \rangle$.

$d_3: \Lambda(\mathcal{J}_2) \rightarrow F$ is a group homomorphism determined by $d_3\langle(1, \beta)\rangle = \langle w_\beta^2 \rangle$.

$d_4: M_4(\mathcal{J}_3) \rightarrow \Lambda$ is a homomorphism determined by $d_4\langle(1, \gamma)\rangle = \langle w_\gamma^3 \rangle$, as described below.

$d_m: M_m(\mathcal{J}_{m-1}) \rightarrow M_{m-1}$ ($4 < m \leq n$) is a module homomorphism determined by $d_m\langle(1, \lambda)\rangle = \langle w_\lambda^{m-1} \rangle$.

We require that all words have trivial boundaries

$$d_m\langle w_\mu^m \rangle = \text{identity } (\mu \in \mathcal{J}_m, 3 \leq m \leq n).$$

If \mathscr{W} is such a word system, the set $\mathscr{W}^{(m)}$ is called the m -field of \mathscr{W} , and its elements w_μ^m are called m -words or words of order m . The boundary homomorphisms d_m ($m > 4$) are well-defined $\mathbf{Z}G$ -maps because M_m is free. The map d_3 coincides with the group homomorphism in the definition of a crossed module. The function $d_4: M_4(\mathcal{J}_3) \rightarrow \Lambda(\mathcal{J}_2)$ is a homomorphism of the underlying groups $d_4: |M_4| \rightarrow |\Lambda|$ that satisfies

$$d_4(g \cdot y) = x \cdot d_4(y)$$

where $y \in M_4$, $g \in G$, and $x \in F$ is any coset representative for $g \in G$. Such a function d_4 is clearly well-defined. When we describe d_4 by the equation $d_4\langle(1, \gamma)\rangle = \langle w_\gamma^3 \rangle$ in Definition (2.1.5), we mean $d_4\langle(g, \gamma)\rangle = x \cdot \langle w_\gamma^3 \rangle$ where $x \in F$ represents the coset $g \in G$.

We often write $\mathscr{W} = \langle \mathscr{W}^{(1)} \mid \mathscr{W}^{(2)} \mid \dots \mid \mathscr{W}^{(n)} \rangle$ in discussions that do not refer to the boundary homomorphisms directly. Such a word system \mathscr{W} is said to have order n . We sometimes write $M_m(\mathscr{W})$ to refer to the module $M_m(\mathcal{J}_{m-1})$ ($m > 3$) or $\Lambda(\mathcal{J}_2)$ ($m = 3$) associated with a word system.

Two word systems

$$\mathscr{W} = \langle \{w_\alpha^1\}_{\alpha \in \mathcal{J}_1} \mid \dots \mid \{w_\nu^n\}_{\nu \in \mathcal{J}_n} \rangle \quad \text{and} \quad \overline{\mathscr{W}} = \langle \{\overline{w}_\alpha^1\}_{\overline{\alpha} \in \overline{\mathcal{J}}_1} \mid \dots \mid \{\overline{w}_\nu^n\}_{\overline{\nu} \in \overline{\mathcal{J}}_n} \rangle$$

have *corresponding index sets* if there are bijections $\mathcal{J}_m \leftrightarrow \overline{\mathcal{J}}_m$ of the index sets for each order $1 \leq m \leq n$. Such word systems are *congruent*, $\mathscr{W} \equiv \overline{\mathscr{W}}$, if they have corresponding index sets and each word \overline{w}_μ^m in $\overline{\mathscr{W}}$ arises from the corresponding word w_μ^m of \mathscr{W} by substituting according to those bijections. We also say that $\overline{\mathscr{W}}$ is obtained from \mathscr{W} by *reindexing*. Congruent word systems are considered to be identical.

2.2 Homotopy reading. The following procedure shows how to associate a word system \mathscr{W} with the structure of any finite connected CW complex K of dimension n . Such a word system \mathscr{W} is called a *homotopy reading* of K .

When $n = 2$, a homotopy reading is a cellular presentation. We also say that K is a *geometric realization* of \mathcal{W} .

1. Identify a maximal tree in K^1 to a point.
2. Select basepoints $*_\mu^m \in \text{bdy}(e_\mu^m)$ for each cell e_μ^m ($\mu \in \mathcal{J}_m$, $2 \leq m \leq n$).
3. Choose connecting paths $\xi_\mu^m: e^0 \rightsquigarrow *_\mu^m$ in K^{m-1} .
4. Choose a characteristic map $\phi_\mu^m: B^m \rightarrow \text{cl}(e_\mu^m)$ for each cell. These determine orientations on the cells.

We may now define Whitehead isomorphisms σ_i as follows:

$\sigma_1: \pi(K^1) \cong F(\mathcal{J}_1)$, $\sigma_1([\phi_\alpha^1]) = \langle \alpha \rangle$, an isomorphism of free groups,
 $\sigma_2: \pi_2(K^2, K^1) \cong \Lambda(\mathcal{J}_2)$, $\sigma_2([\xi_\beta^2] \cdot [\phi_\beta^2]) = \langle (1, \beta) \rangle$, so that (σ_1, σ_2) is an isomorphism of free crossed modules, and

$\sigma_m: \pi_m(K^m, K^{m-1}) \cong M_{m+1}(\mathcal{J}_m)$, $\sigma_m([\xi_\mu^m] \cdot [\phi_\mu^m]) = \langle (1, \mu) \rangle$, an isomorphism of free modules ($3 \leq m \leq n$).

5. Let w_μ^m be any word representing the element
 (2.2.1)

$$\langle w_\mu^m \rangle = \begin{cases} \sigma_1([\xi_\mu^2 \# \partial \phi_\mu^2 \# \overline{\xi_\mu^2}]) \in F(\mathcal{J}_1) & (\mu \in \mathcal{J}_2, m = 2), \\ \sigma_{m-1}([\xi_\mu^m] \cdot [\phi_\mu^m]) \in M_m(I_{m-1}) \text{ or } \Lambda(\mathcal{J}_2) & (\mu \in \mathcal{J}_m, 3 \leq m \leq n). \end{cases}$$

6. Finally, define boundary maps $d_{m+1}: M_{m+1}(\mathcal{J}_m) \rightarrow M_m(\mathcal{J}_{m-1})$ by $d_{m+1}\langle (1, \mu) \rangle = \langle w_\mu^m \rangle$, where M_3 and M_2 are understood to mean Λ and F , respectively.

The following relations are easily checked (∂_m is the usual boundary operator in relative homotopy).

(2.2.2)

$$\begin{aligned} d_3 \circ \sigma_2 &= \sigma_1 \circ \partial_2: \pi_2(K^2, K^1) \rightarrow F(\mathcal{J}_1), \\ d_4 \circ \sigma_3 &= \sigma_2 \circ \partial_3: \pi_3(K^3, K^2) \rightarrow \Lambda(\mathcal{J}_2), \\ d_{m+1} \circ \sigma_m &= \sigma_{m-1} \circ \partial_m: \pi_m(K^m, K^{m-1}) \rightarrow M_m(\mathcal{J}_{m-1}) \quad (4 \leq m \leq n-1). \end{aligned}$$

Thus

$$d_m \langle w_\mu^m \rangle = d_m d_{m+1} \sigma_m([\xi_\mu^m] \cdot [\phi_\mu^m]) = \sigma_{m-2} \partial_{m-1} \partial_m([\xi_\mu^m] \cdot [\phi_\mu^m]) = 0 \in M_{m-1}.$$

This verifies that $\mathcal{W} = (\langle \{w_\alpha^1\} \mid \{w_\beta^2\} \mid \cdots \mid \{w_\nu^n\} \rangle, \{d_m\})$ is a word system.

The correspondence between a homotopy reading and a geometric realization is completely determined by the choices 1 to 5 above. We refer to these selections as *homotopy reading choices*. If a word w and an attaching map $f = \partial \phi$ are related according to (2.2.1) above, we say that w is *associated with* f , $w \sim f$.

As was noted in Subsection 1.3, every group presentation P is a cellular presentation of some 2-complex $K(\mathcal{P})$. Thus, every word system $\mathcal{W} = \langle \mathcal{W}^{(1)} \mid \mathcal{W}^{(2)} \rangle$ of order 2 has a geometric realization. However, Whitehead's study of homotopy systems shows that higher-order words systems may fail to have geometric realizations.

Example (2.2.3) [14, §15]. The word system

$$\begin{aligned} \mathcal{W} &= \langle x \mid w_a^2, w_b^2 \mid w_c^3 \mid w_d^4 \mid w_e^5 \rangle, \\ w_a^2 &= xx, \quad w_b^2 = 1, \quad w_c^3 = (\langle x \rangle, a)(1, a)^{-1}, \\ w_d^4 &= (\bar{x}, c) + (1, c), \quad w_e^5 = (\bar{x}, d) - (1, d) \end{aligned}$$

has no geometric realization, where \bar{x} is the element in $G = F/R$ represented by $\langle x \rangle \in F$.

Such “phantom complexes” arise when a boundary condition $d_n(\langle w^n \rangle) = 0$ corresponds to a triviality in relative homotopy that is not a triviality in absolute homotopy (cf. (2.2.2) and Proposition (3.2.3)). Whitehead’s work implies the following results about geometric realization.

Proposition (2.2.4) [14, Theorem 2, p. 465]. *Every word system of order $n \leq 4$ has a geometric realization.* \square

Proposition (2.2.5) [14, Theorem 3, p. 466]. *If \mathcal{W} is a word system for which the group $G(\mathcal{W})$ presented by $\langle \mathcal{W}^{(1)} \mid \mathcal{W}^{(2)} \rangle$ is trivial, then \mathcal{W} has a geometric realization.* \square

2.3 Equivalence relations. Let \mathcal{W} and $\overline{\mathcal{W}}$ be word systems. Denote the words, index sets and boundary maps of \mathcal{W} by w_μ^m , \mathcal{J}_m , and d_m , and denote those constituents of $\overline{\mathcal{W}}$ by \overline{w}_μ^m , $\overline{\mathcal{J}}_m$, and \overline{d}_m .

In Definition (2.3.5), we introduce three equivalence relations on word systems in the spirit of the EN-equivalence relation on presentations. Some complications arise because moves such as product equivalence of low-order words may lead to spelling changes in higher-order words. N.B. For convenience, we use the additive notation (2.1.2) here even in the case $m = 3$.

Definition (2.3.1). The move $[\mathcal{W}, \overline{\mathcal{W}}]$ is an *elementary 1-product equivalence* if the following conditions hold for some 1-indices $\alpha, \tau \in \mathcal{J}_1$ and $\bar{\alpha} \in \overline{\mathcal{J}}_1$.

- (1) $\overline{\mathcal{J}}_1 = \mathcal{J}_1 \cup \{\bar{\alpha}\} - \{\alpha\}$, and $\overline{\mathcal{J}}_m = \mathcal{J}_m$ ($m \geq 2$).
- (2) Each 2-word $\overline{w}_\beta^2 \in \overline{\mathcal{W}}$ is obtained from the corresponding 2-word $w_\beta^2 \in \mathcal{W}$ by replacing each occurrence of α by $\bar{\alpha}\tau^{-1}$.
- (3) For each m -word $w_\mu^m = \varepsilon_1(g_1, \lambda_1) + \cdots + \varepsilon_Q(g_Q, \lambda_Q) \in \mathcal{W}$ ($m \geq 3$), the corresponding word $\overline{w}_\mu^m \in \overline{\mathcal{W}}$ has the form $\overline{w}_\mu^m = \varepsilon_1(\bar{g}_1, \lambda_1) + \cdots + \varepsilon_Q(\bar{g}_Q, \lambda_Q)$, where a representing word for the element \bar{g}_q arises from a representing word for the element g_q by replacing each occurrence of α by $\bar{\alpha}\tau^{-1}$.

Definition (2.3.2). For $m > 1$, the move $[\mathcal{W}, \overline{\mathcal{W}}]$ is an *elementary m -product equivalence* if the following conditions hold for m -indices $\mu, \tau \in \mathcal{J}_m$ and $\bar{\mu} \in \overline{\mathcal{J}}_m$.

- (1) $\overline{\mathcal{J}}_m = \mathcal{J}_m \cup \{\bar{\mu}\} - \{\mu\}$, and $\overline{\mathcal{J}}_j = \mathcal{J}_j$ ($j \neq m$).
- (2) $\overline{w}_\mu^m = w_\mu^m + w_\tau^m$.
- (3) Each $(m+1)$ -word $\overline{w}_\xi^{m+1} \in \overline{\mathcal{W}}$ ($\xi \in \mathcal{J}_{m+1}$) arises from $w_\xi^{m+1} \in \mathcal{W}$ by replacing every μ -syllable $\varepsilon_q(g_q, \mu)$ by $\varepsilon_q(g_q, \bar{\mu}) - \varepsilon_q(g_q, \tau)$.
- (4) $\overline{w}_\sigma^p = w_\sigma^p$ for all $p, \sigma \in \mathcal{J}_p$ except for the cases above.

Definition (2.3.3). Suppose $m > 1$. The move $[\mathcal{W}, \overline{\mathcal{W}}]$ is an *elementary m -expansion*, $\mathcal{W} \nearrow_{e,m} \overline{\mathcal{W}}$, if there are an m -index $\mu \in \mathcal{J}_m$ and an $(m-1)$ -index $\lambda \in \mathcal{J}_{m-1}$ such that

- (1) λ occurs exactly once in w_μ^m , and
- (2) \mathcal{W} is congruent to $\overline{\mathcal{W}} - \{w_\mu^m, w_\lambda^{m-1}\}$.

We also say that $[\overline{\mathcal{W}}, \mathcal{W}]$ is an *elementary m -collapse*, $\overline{\mathcal{W}} \searrow_{e,m} \mathcal{W}$. The index λ is called the *expansion face index* (or *collapse face index*).

Definition (2.3.4). The *elementary P_n -moves* are:

- P_n1 . *Word-representative equivalence.* Replace one word w_μ^m ($m \geq 2$) by a word \overline{w}_μ^m such that $\langle w_\mu^m \rangle = \langle \overline{w}_\mu^m \rangle \in M_m$.
- P_n2 . *Conjugation.* Let $\alpha \in \mathcal{J}$. If $m = 2$, replace w_β^2 by $\alpha w_\beta^2 \alpha^{-1}$. If $m > 2$, replace one word w_μ^m by $\langle \alpha \rangle \cdot w_\mu^m$. Make appropriate adjustments in higher order words.
- P_n3 . *Inversion.* Replace one word w_μ^n ($m \geq 2$) by the inverted word $(w_\mu^m)^{-1}$ ($m = 2, 3$) or $-w_\mu^m$ ($m > 3$). Make appropriate adjustments in higher order words.
- P_n4 . *Elementary m -expansion or m -collapse*, where $2 \leq m \leq n$ (2.3.3).
- P_n5 . *Elementary n -product equivalence* (2.3.1) or (2.3.2).

Definition (2.3.5). Let \mathcal{W} and $\overline{\mathcal{W}}$ be two word systems.

- (a) \mathcal{W} and $\overline{\mathcal{W}}$ are *P_n -equivalent* if there is a sequence of elementary P_n -moves that transforms \mathcal{W} to $\overline{\mathcal{W}}$.
- (b) \mathcal{W} and $\overline{\mathcal{W}}$ are *fully P_n -equivalent* if $\overline{\mathcal{W}}$ can be obtained from \mathcal{W} by a finite sequence of moves of types P_n1, \dots, P_n4 and elementary m -product equivalences of any order $1 \leq m \leq n$.
- (c) \mathcal{W} and $\overline{\mathcal{W}}$ are *D_n -equivalent* if $\overline{\mathcal{W}}$ can be obtained from \mathcal{W} by a finite sequence of moves of types P_n3 and P_n4 only.

These equivalence relations are summarized in Table (2.3.6).

TABLE (2.3.6). Equivalence relations among word systems

| Move | Orders m permitted for | | |
|---|--------------------------|-------------------|-------------------|
| | P_n | full P_n | D_n |
| word-representative | $2 \leq m \leq n$ | $2 \leq m \leq n$ | — |
| conjugation | $2 \leq m \leq n$ | $2 \leq m \leq n$ | — |
| inversion | $2 \leq m \leq n$ | $2 \leq m \leq n$ | $2 \leq m \leq n$ |
| elementary expansion | $2 \leq m \leq n$ | $2 \leq m \leq n$ | $2 \leq m \leq n$ |
| elementary collapse | $2 \leq m \leq n$ | $2 \leq m \leq n$ | $2 \leq m \leq n$ |
| produce equivalence | $m = n$ | $1 \leq m \leq n$ | — |
| Note: Reindexing yields congruent word systems, which are considered the same | | | |

A sequence of word systems $\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_K$ will be called a *P_n -sequence* if each elementary move $[\mathcal{W}_k, \mathcal{W}_{k+1}]$ is a P_n -move. We define *full P_n -sequence* and *D_n -sequence* similarly. If every such move is an elementary expansion or collapse, we write $\mathcal{W}_1 \nearrow \mathcal{W}_K$. In that case, the notations $\mathcal{W}_1 \nearrow \mathcal{W}_K$, $\mathcal{W}_1 \searrow \mathcal{W}_K$, $\mathcal{W}_k \nearrow \mathcal{W}_K$, etc., are used by analogy with formal deformations of CW complexes. If \mathcal{W}_1 and \mathcal{W}_K have order n , we say that a D_{n+1} -sequence $\mathcal{W}_1, \dots, \mathcal{W}_K$ is *transient* if every $(n+1)$ -word is collapsed immediately after it is expanded.

Lemma (2.3.7). *Suppose that K is a CW complex. Let \mathcal{W} be a homotopy reading of K for which a maximal tree $T \subset K^1$ is identified, and let \bar{T} be another maximal tree in K^1 .*

Then there is a homotopy reading $\overline{\mathcal{W}}$ of K for which \bar{T} is identified such that \mathcal{W} can be transformed into $\overline{\mathcal{W}}$ by a finite sequence of 1-product equivalences and 1-inversions.

Furthermore, each index α occurs the same number of times in each word $w \in \mathcal{W}$ as in the corresponding word $\bar{w} \in \overline{\mathcal{W}}$, unless α is a 1-index represented in one tree and not in the other.

Proof. Generalize [11, 4.6] to word systems. \square

Proposition (2.3.8). *If \mathcal{W} and $\overline{\mathcal{W}}$ are two homotopy readings for the same complex K , then \mathcal{W} can be transformed to $\overline{\mathcal{W}}$ using word-representative equivalence, conjugation, inversion and elementary 1-product equivalence.*

Proof. By the lemma, we may assume the same choice of maximal tree for both readings. We consider the remaining homotopy reading choices of Subsection 2.2. Changes of cell basepoints lead to conjugations. Changes of connecting paths also lead to conjugations. Proposition (1.2.1) implies that we may assume the same characteristic maps after conjugations and inversions. Word-representative equivalence allows us to choose the same words for both homotopy readings. Thus, we can transform \mathcal{W} to a word system \mathcal{W}' that is a homotopy reading for K using all of the same choices as for $\overline{\mathcal{W}}$. It follows that $\mathcal{W}' = \overline{\mathcal{W}}$. \square

3. PROPERTIES OF HOMOTOPY READING

3.1 Multiple collapse maps and geometric words. If e^m is an m -cell that can be collapsed in several different ways, we want to show there exists a word w^m associated with e^m that expresses all of the collapse conditions simultaneously. Such a word w^n is called a *geometric word*, and can be “read” from a multiple collapse map. The existence of multiple collapse maps for 3-cells requires special attention (Proposition (3.1.2)), and implies a claim made without proof in [16, p. 24]. We give a decomposition space argument for that case.

We begin with a lemma.

Lemma (3.1.1). *Let X and Y be topological spaces. Suppose that the function $p: X \rightarrow Y$ is either (a) a quotient map or (b) a characteristic map for a cell in Y . Let $h: X \rightarrow Z$ be a continuous function such that $h \circ p^{-1}$ is single valued.*

Then $h \circ p^{-1}$ is continuous. Furthermore, if h is closed then $h \circ p^{-1}$ is closed.

Proof. This is an application of transgression [5, Chapter VI, 3.2]. \square

Proposition (3.1.2) (Existence of multiple collapse maps in dimension three). *Let e^3 be a 3-cell and let $e_1^2, e_2^2, \dots, e_j^2$ be 2-cells in a CW complex K . Suppose that collapse maps exist for each of the pairs $(e^3, e_1^2), (e^3, e_2^2), \dots, (e^3, e_j^2)$.*

Then a multiple collapse map exists for $(e^3, e_1^2, \dots, e_j^2)$.

Proof. Inductively assume that $\phi_j: E^3 \rightarrow \text{cl}(e^3)$ is a multiple collapse map for $(e^3, e_1^2, \dots, e_j^2)$. We will argue that a multiple collapse map ϕ_{j+1} for $(e^3, e_1^2, \dots, e_{j+1}^2)$ can be obtained from ϕ_j by modifying it near the preimage

$\phi_j^{-1}(e_{j+1}^2)$, then forming a decomposition space E^3/G of E^3 . The nondegenerate elements of the decomposition G will be point-inverses $\phi_j^{-1}(x)$ of certain elements $x \in e_{j+1}^2$.

Let S be the preimage $S = \phi_j^{-1}(e_{j+1}^2) \subset E^3$. The following conditions clearly hold.

Condition (3.1.3). *For each point $x \in e_{j+1}^2$ the point-inverse $\phi_j^{-1}(x)$ is connected and does not separate the boundary sphere $\text{bdy}(E^3)$.*

We now seek to modify the characteristic map ϕ_j near S in order to bypass potential complications at the frontier $\text{fr}(S)$.

Let $\Phi: D^3 \rightarrow \text{cl}(e^3)$ be a collapse map for (e^3, e_{j+1}^2) as guaranteed by the hypothesis, defined on a 3-ball D^3 . By definition, the map Φ restricts to a characteristic map $\rho: D^2 \rightarrow \text{cl}(e_{j+1}^2)$. The 2-ball $D^2 \subset \text{bdy}(D^3)$ may be identified with the unit disk $\{(r, \theta) | 0 \leq r \leq 1\}$ in \mathbf{R}^2 . Extend the continuous function $\lambda: D^2 \rightarrow D^2$ defined by

$$\lambda(r, \theta) = \begin{cases} (2r, \theta) & \text{if } r \leq \frac{1}{2}, \\ (1, \theta) & \text{if } r \geq \frac{1}{2}, \end{cases}$$

to a continuous function $\Lambda: D^3 \rightarrow D^3$ that is a homeomorphism on $\text{int}(D^3)$ such that $\Lambda|(\text{bdy}(D^3) - \text{int}(D^2)) = \text{identity}$. Then the composition $H = \Phi \circ \Lambda \circ \Phi^{-1}: \text{cl}(e^3) \rightarrow \text{cl}(e^3)$ is a well-defined continuous map satisfying

$$H|(\text{bdy}(e^3) - e^2) = \text{identity}.$$

It follows that there is a well-defined continuous map $\tilde{\phi}_j = H \circ \phi_j: E^3 \rightarrow \text{cl}(e^3)$. Because of the nature of H , this map $\tilde{\phi}_j$ is seen to be a multiple collapse map for $(e^3, e_1^2, \dots, e_j^2)$.

Let G be the decomposition of R^3 whose nondegenerate elements are the point inverses $f_j^{-1}(x)$ for points $x \in A^2$, where A^2 is the ball

$$A^2 = \rho(\{(r, \theta) | r \leq \frac{1}{2}\}) \subset e_{j+1}^2.$$

(Here we emphasize that f_j is the *original* attaching map $\phi_j| \text{bdy}(E^3)$.) Let $q: E^3 \rightarrow E^3/G$ be the associated quotient map. Then G is an upper semicontinuous decomposition of the separable metric space E^3 , so E^3/G is a metric space by a well-known theorem. Furthermore, E^3/G is a Hausdorff space, and E^3 is compact, so q is a closed map. Recalling Condition (3.1.3), we may conclude that E^3/G is a 3-ball by [6, Theorem 8]. Now define the function $\phi_{j+1}: E/G \rightarrow \text{cl}(e^3)$ by $\phi_{j+1}(\bar{x}) = \tilde{\phi}_j(x)$ where \bar{x} is the decomposition element containing x . It is easy to show that ϕ_{j+1} is well defined. Lemma (3.1.1) implies that the function $\phi_{j+1} = \tilde{\phi}_j \circ q^{-1}$ is continuous and closed. In addition, $\phi_{j+1}: E^3/G \rightarrow \text{cl}(e^3)$ is clearly a surjection, and $\phi_{j+1}|(\text{int}(E^3)/G)$ is a homeomorphism $\text{int}(E^3)/G \approx \text{int}(e^3)$ because this restriction is a bijection. Therefore ϕ_{j+1} restricts to characteristic maps for e_1^2, \dots, e_j^2 since the decomposition G is trivial away from S . It suffices to show that ϕ_{j+1} restricts to a characteristic map for e_{j+1}^2 .

The subspace $b = f_j^{-1}(A^2)/G \subset E^3/G$ is a closed 2-ball, since G consists exactly of the nondegenerate point-inverses of the continuous closed surjection

$f_j|f_j^{-1}(A^2): f_j^{-1}(A^2) \rightarrow A^2$ onto the 2-ball A^2 . Furthermore, the continuous closed surjection ϕ_{j+1} is bijective on $\text{int}(b)$ by construction, so $\phi_{j+1}|_{\text{int}(b)}$ is a homeomorphism $\text{int}(b) \approx e_{j+1}^2$. Thus $\phi_{j+1}|_b$ is a characteristic map for e_{j+1}^2 . \square

We next check that if w is an m -word representing a map in a CW m -complex X , then w also represents the induced map after more m -cells are added to X .

Proposition (3.1.4). *Let $m \geq 2$ and suppose that K is an m -complex with m -cells $\{e_\mu^m \mid \mu \in \mathcal{J}_m\}$. Let \mathcal{A} and \mathcal{B} be disjoint subsets of \mathcal{J}_m satisfying $\mathcal{A} \cup \mathcal{B} = \mathcal{J}_m$, and define subcomplexes $X = K^{m-1} \cup \{e_\alpha^m \mid \alpha \in \mathcal{A}\}$ and $Y = K^{m-1} \cup \{e_\beta^m \mid \beta \in \mathcal{B}\}$.*

Then $\pi_m(K^m, K^{m-1})$ is the free product of $\pi_m(X, K^{m-1})$ and $\pi_m(Y, K^{m-1})$ in the category of $\pi_1(K^{m-1})$ -crossed modules.

Proof. This follows from the generalized Seifert-Van Kampen Theorem of [3, Theorem C]. \square

Corollary (3.1.5). *With the hypotheses of Proposition (3.1.4), suppose that \bar{w} is a word associated with a map $f: B^m \rightarrow X$ via an isomorphism*

$$\sigma_{m,X}: \pi_m(X, K^{m-1}) \rightarrow M_{m+1}(\mathcal{A})$$

(as in Subsection 2.2). Suppose that the isomorphism $\sigma_m: \pi_m(K^m, K^{m-1}) \rightarrow M_{m+1}(\mathcal{J}_m)$ satisfies $\sigma_m \circ i_ = \sigma_{m,X}$, where $i_*: \pi_m(X, K^{m-1}) \rightarrow \pi_m(K^m, K^{m-1})$ is induced by inclusion.*

Then that same word \bar{w} is also associated with the map $\text{incl} \circ f: B^m \rightarrow K$, that is, $\sigma_m([\text{incl} \circ f]) = \langle \bar{w} \rangle \in M_{m+1}(\mathcal{J}_m)$. \square

Proposition (3.1.6) (Geometric reading of multiple collapse maps). *Let $m \geq 3$, and let e_μ^m be a cell in a CW complex K . Suppose that a multiple collapse map $\phi_\mu^m: B \rightarrow \text{cl}(e_\mu^m)$ exists for $(e_\mu^m, e_{\alpha_1}^{m-1}, \dots, e_{\alpha_Q}^{m-1})$.*

Then there is a word $w \sim \partial \phi_\mu^m$ that mentions each $(m-1)$ -index α_q exactly once.

Proof. Define subcomplexes $X = K^{m-2} \cup \{e_\alpha^{m-1} \mid \alpha \in \mathcal{A}\}$ and $Y = K^{m-2} \cup \{e_\beta^{m-1} \mid \beta \in \mathcal{B}\}$, where $\mathcal{A} = \{\alpha_1, \dots, \alpha_Q\}$ and $\mathcal{B} = \mathcal{J}_m - \mathcal{A}$.

Let (B, b_1, \dots, b_Q) be a topological m -ball tuple for ϕ_μ^m such that $\phi_\mu^m|_{b_q}$ is a characteristic map for $e_{\alpha_q}^{m-1}$ ($1 \leq q \leq Q$), as guaranteed in the definition of a multiple collapse map. Connect the $(m-1)$ -balls b_q to the basepoint of B by nonintersecting arcs $a_q: I \rightarrow (\phi_\mu^m)^{-1}(K^{m-2})$. Fatten these arcs slightly to form a closed neighborhood N about $\bigcup_q \{a_q \cup b_q\}$ so that both N and $\text{cl}(B - N)$ are topological $(m-1)$ -balls. We may modify the attaching map f_μ^m by a homotopy $f_\mu^m \simeq f_1 \text{ (rel } *_B)$ so that the image $f_1(N - \bigcup_q b_q)$ is contained in K^{m-2} . By Proposition (1.2.1), the map f_1 is homotopic to a map f_2 such that all of the following conditions hold.

- (a) For some interior ball $b'_q \subset \text{int}(b_q)$, the restriction $f_2|_{b'_q}$ is a copy of the characteristic map $\phi_{\alpha_q}^2$, up to reversal.
- (b) $f_2(N - \bigcup_q (b'_q)) \subset K^{m-2}$.
- (c) $f_2|_{\text{cl}(\dot{B} - N)} = f_1|_{\text{cl}(\dot{B} - N)}$, where $\dot{B} = \text{bdy } B$.

By the definition of the Whitehead isomorphism $\sigma_{m-1}: \pi_{m-1}(X, K^{m-2}) \rightarrow M_m(\mathcal{A})$ of Theorem (2.1.3) or Theorem (2.1.4), the map $f_2|N: N \rightarrow X$ is associated with a word $w_1 = \varepsilon_1(g_1, \alpha_1) + \cdots + \varepsilon_q(g_q, \alpha_q)$, i.e.,

$$\langle w_1 \rangle = \sigma_{m-1, X}([f_2|N]) \in M_m(\mathcal{A}).$$

(When $m = 3$, $w_1 = ((g_1, \alpha_1)^{\varepsilon_1} \cdots (g_q, \alpha_q)^{\varepsilon_q})$. Also, $f_2(\text{cl}(\dot{B} - N))$ is contained in the subcomplex Y and has an associated word w_2 representing an element in $M_m(\mathcal{B})$. Such a word w_2 cannot possibly mention any indices $\alpha_1, \dots, \alpha_Q$.)

Applying Corollary (3.1.5) to the words w_1 and w_2 , we see that

$$\sigma_{m-1}([f_\mu^m]) = \sigma_{m-1}([f_2]) = \begin{cases} \langle w_1 \rangle + \langle w_2 \rangle = \langle w_1 + w_2 \rangle & \text{when } m > 3, \\ \langle w_1 \rangle \langle w_2 \rangle = \langle w_1 w_2 \rangle & \text{when } m = 3. \end{cases}$$

Let $w = w_1 + w_2$ if $m > 3$, or $w = w_1 w_2$ if $m = 3$. Then $w \sim \partial \phi_\mu^m = f_\mu^m$, where w is a word that mentions each $(m-1)$ -index $\alpha_1, \dots, \alpha_Q$ exactly once. \square

Suppose that e^m is an m -cell ($m \geq 2$) in a CW complex K , and that $(e^m, e_{\alpha_1}^{m-1}), \dots, (e^m, e_{\alpha_Q}^{m-1})$ is a complete set of pairs for which collapse maps exist. Let σ_{m-1} be a Whitehead isomorphism as in Theorem (2.1.3) or Theorem (2.1.4). A word w^m is a *geometric word* for e^m if

- (a) $w^m \sim f^m$,
- (b) w^m mentions the $(m-1)$ -indices $\alpha_1, \dots, \alpha_Q$ exactly once each, and
- (c) all $(m-1)$ -indices α mentioned in w^m are indices of $(m-1)$ -cells e_α^{m-1} contained in the carrier subcomplex for e^m (i.e., the smallest subcomplex of K containing $\text{cl}(e^m)$).

N.B. When the dimension is $m = 2$, condition (b) applies only to 1-indices α_q for which $e_{\alpha_q}^1$ is not part of the maximal tree $T \subset K^1$ chosen for the homotopy reading, since 1-cells in T were not assigned indices in \mathcal{J}_1 .

Proposition (3.1.7) (Existence of geometric words). *A geometric word w^m exists for every cell e^m of dimension $m \geq 2$ in a CW complex K .*

Proof. The argument is straightforward when $m = 2$. For $m > 3$, let α_q be among the indices $\alpha_1, \dots, \alpha_Q$ referred to in the definition of a geometric word. Proposition (3.1.6) shows that there is a word $w_q^m \sim f^m$ that mentions the index α_q exactly once. It follows that any freely reduced word $w^m \sim f^m$ mentions the index α_q exactly once, for all q ($1 \leq q \leq Q$). We may assume that w^m satisfies the carrier condition (c) by Corollary (3.1.5), so w^m is a geometric word for e^m .

When the dimension is $m = 3$, freely reduced words are not available. The proposition follows in this case by applying geometric reading (3.1.6) and Corollary (3.1.5) to a multiple collapse map for e^m , which exists by Proposition (3.1.2). \square

Corollary (3.1.8). *Suppose that $K \nearrow_{e, m} \bar{K}$ where K and \bar{K} are CW complexes, and that \mathcal{W} is a homotopy reading for K consisting entirely of geometric words. Then $\mathcal{W} \nearrow_{e, m} \bar{\mathcal{W}}$, where $\bar{\mathcal{W}}$ is a homotopy reading for \bar{K} consisting entirely of geometric words. \square*

3.2 Model cells and model complexes. Having found words that reflect the collapse structure of cells, we now seek complexes that geometrically reflect the word structure of a word system. Such specialized complexes will be used in Proposition (3.3.5) to show that any two complexes sharing the same homotopy reading must homotopy n -deform to each other.

Suppose that K is a CW complex with one 0-cell e^0 . A 2-cell $e^2 \in K$ is a *model cell* for $w^2 = \alpha_1^{\varepsilon_1} \cdots \alpha_Q^{\varepsilon_Q}$ if there is a subdivision of $S^1 = \text{bdy}(E^2)$ into consecutive arcs A_1, \dots, A_Q such that $\phi^2|_{A_q}$ is a copy of the characteristic map $\phi_{\alpha_q}^1$ for $e_{\alpha_q}^1$, with orientation determined by ε_q .

We say that a path $\gamma: \mathbf{I} \rightarrow K^1$ *models an element* $x \in F(\mathcal{J}_1)$ if the condition $\phi_1([\gamma]) = x$ holds. We also say that $\gamma: \mathbf{I} \rightarrow K^1$ *models an element* $g \in G = F(\mathcal{J}_1)/R$ if γ models a coset representative x of g .

An m -cell $e^m \in K$ ($m \geq 3$) with characteristic map $\phi^m: E^m \rightarrow \text{cl}(e^m)$ is a *model cell* for $w^m = \varepsilon_1(x_1, \lambda_1) + \cdots + \varepsilon_Q(x_Q, \lambda_Q)$ if there exist disjoint closed $(m-1)$ -balls B_1, \dots, B_Q in $\text{bdy}(E^m)$ such that the following conditions hold.

- (1) $\phi^m(\text{bdy}(E^m) - (\bigcup B_q)) \subset K^{m-2}$.
- (2) $\phi^m|_{B_q}$ is a copy of the characteristic map $\phi_{\lambda_q}^{m-1}$ for $e_{\lambda_q}^{m-1}$, with orientation determined by ε_q .
- (3) There are paths $\alpha_q: (*_{E^m}) \rightsquigarrow (*_{B_q})$ in $\text{bdy}(E^m)$ such that
 - (a) the union of the images of the paths α_q is a tree that meets each ball B_q in a unique extreme point, and
 - (b) $\phi^m \circ \alpha_q$ is a path product $\phi^m \circ \alpha_q = \gamma_q \# \xi_{\lambda_q}^{m-1}$, where $\gamma_q: \mathbf{I} \rightarrow K^1$ models x_q and $\xi_{\lambda_q}^{m-1}$ is a fixed choice of connecting path $\xi_{\lambda_q}^{m-1}: (e^0) \rightsquigarrow (*_{\lambda_q}^{m-1})$ for each cell $e_{\lambda_q}^{m-1}$.

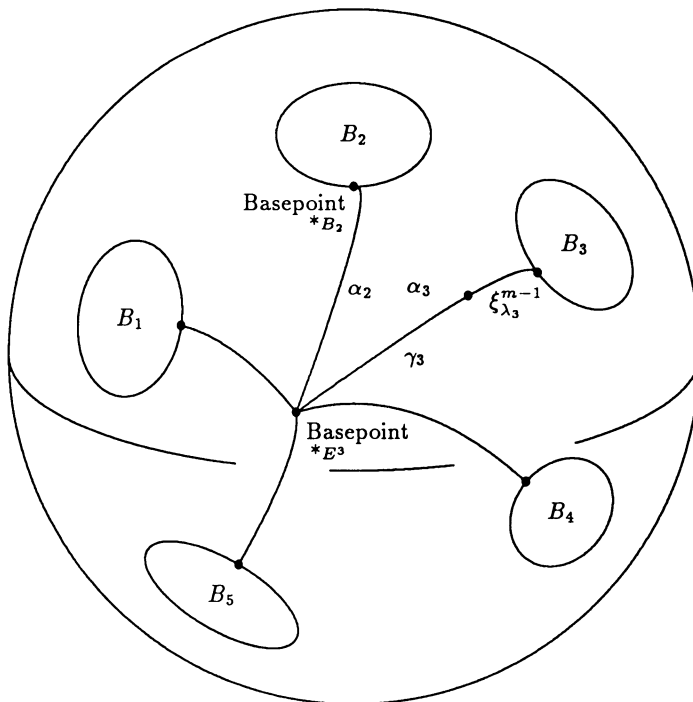


FIGURE (3.2.1). Diagram of a model cell for $m = 3$

When the dimension is $m = 3$, we further require that the balls B_q be encountered in the natural order B_1, \dots, B_Q as one traverses the tree system around the basepoint of E^3 (see Figure (3.2.1)).

The attaching map $f^m = \phi^m|_{\text{bdy}(E^m)}$ of such a cell is called a *model map* for w^m . It is clear that $w^m \sim f^m$.

A CW complex is a *model complex* $K(\mathcal{W})$ for a word system \mathcal{W} if $K(\mathcal{W})$ has one 0-cell and each cell $e_\mu^m \in K(\mathcal{W})$ is a model cell for the corresponding word $w_\mu^m \in \mathcal{W}$. Trivially, a model complex $K(\mathcal{W})$ is a geometric realization of \mathcal{W} .

Proposition (3.2.2). *If f is a model map for a word w^m then f is also a model map for any cyclic permutation of w^m . \square*

Obviously, model maps exist for any 2-word.

Proposition (3.2.3) (Existence of model maps). *Let \mathcal{W} be a homotopy reading of a CW complex K . Let w be a word, not necessarily in \mathcal{W} , that represents an element in $M_m(\mathcal{W})$ ($m > 2$). Let*

$$f: (I^{m-1}, \text{bdy}(I^{m-1}), I^{m-2} \times 0) \rightarrow (K^{m-1}, K^{m-2}, *)$$

be a map such that $w \sim f$ and $\beta([f]) = 0$, for the homotopy boundary homomorphism $\beta: \pi_{m-1}(K^{m-1}, K^{m-2}) \rightarrow \pi_{m-2}(K^{m-2})$.

Then f is homotopic rel basepoint in K^{m-1} to a model map f^ for w .*

In particular, if $f = f^m$ is an attaching map for a cell, then a model map exists for any word $w \sim f^m$.

Remarks. (1) Observe that $d_m(\langle w \rangle) = 1$ follows from the conditions on f . (2) We may not assume *a priori* that a map such as f exists when $m > 4$, as demonstrated by the word w_e^5 of Example (2.2.3) for which $\beta([f]) = 0$ cannot be satisfied.

Proof of Proposition (3.2.3). Write w in the form $w = \varepsilon_1(g_1, \lambda_1) + \dots + \varepsilon_Q(g_Q, \lambda_Q)$. Let X be a subspace of the $(m-1)$ -sphere S^{m-1} consisting of $(m-1)$ -balls B_1, \dots, B_Q and simple, nonintersecting arcs α_q joining the basepoint of S^{m-1} to the basepoint of B_q ($1 \leq q \leq Q$). Let N be a neighborhood of X found by thickening $\text{fr}(X)$, such that N and $\text{cl}(S^{m-1} - N)$ are $(m-1)$ -balls.

Let $F: X \rightarrow K^{m-1}$ be a map such that $F|_{B_q}$ is a copy of the characteristic map $\phi_{\lambda_q}^{m-1}$ for some path γ_q that models the element g_q . Extend F to a map $\tilde{F}: (N, \text{bdy}(N)) \rightarrow (K^{m-1}, K^{m-2})$ by means of a retraction r , so that \tilde{F} is associated with the element $\langle w \rangle$. This means that $\sigma_{m-1}([\tilde{F}]) = \langle w \rangle \in M_m(\mathcal{W})$, where σ_{m-1} is a Whitehead isomorphism.

A computation using the hypothesis $[f] = \sigma^{-1}(\langle w \rangle) = [\tilde{F}]$ and $\beta([f]) = 0 \in \pi_{m-2}(K^{m-2})$ shows that $\partial \tilde{F}$ is nullhomotopic. Use this nullhomotopy to extend \tilde{F} to a map $F^*: S^{m-1} \rightarrow K^{m-1}$. F^* is clearly a model map for w , and $\sigma_{m-1}([F^*]) = \langle w \rangle = \sigma_{m-1}([\tilde{F}])$ implies that $[F^*] = [f] \in \pi_{m-1}(K^{m-1}, K^{m-2})$. \square

3.3 Geometric dragging and substitution. Let K be an n -complex. Geometric dragging is a technique by which an elementary homotopy m -deformation of the

m -skeleton of K ($m \leq n$) is lifted to a homotopy n -deformation $K \xrightarrow{n} L$ of the entire complex K . Our terminology is motivated by the following example.

Example (3.3.1). Let $D = e^0 \cup e^1 \cup e^2$ be a CW decomposition of a disk, and consider the subcomplex $K_1 = (e^0 \cup e^1) \times I$ of $D \times I$. Consider the elementary homotopy 2-move

$$K = K_1 \cup (e^2 \times 0) \xrightarrow{e, 2} K_1 \cup (e^2 \times 1) = L$$

and let e^3 be the 3-cell of the associated 3-expansion.

Now suppose that K is a subcomplex of a CW complex X that does not include e^3 , and let e_0^3 be a 3-cell in X that has the 2-cell $e^2 \times 0$ in its boundary. The homotopy move $K \xrightarrow{e, 2} L$ above could be used to induce a modification of

the attaching map of e_0^3 , yielding a new 3-cell e_1^3 that has the 2-cells $e^2 \times 1$ and $e^1 \times (0, 1)$ in its boundary. We describe this as “dragging” e_0^3 to e_1^3 .

The effect of that homotopy move on a word w_0^3 for e_0^3 can be described as substitution. For example, suppose that $w^3 = (1, \mu_0)^{-1}(1, \mu)(1, \mu_1)$ is a geometric word for e^3 in some homotopy reading of $X \cup (D \times I)$, where $\mu_i \in \mathcal{J}_2$ is the index for $e^2 \times i$ ($i = 0, 1$) and $\mu \in \mathcal{J}_2$ is the index for $e^1 \times (0, 1)$. Then under suitable conditions on e_0^3 , every occurrence of the syllable $(1, \mu_0)$ in w_0^3 may be replaced by the word $\hat{w} = (1, \mu)(1, \mu_1)$ to yield a word w_1^3 for e_1^3 . \square

Proposition (3.3.2) (Internal collapse of a maximal tree). *Let $T \subset K^1$ be a maximal tree in an n -complex K .*

Then $K \xrightarrow{n} K/T$.

Proof. Let $F: K \rightarrow K/T$ be the quotient map, and consider the mapping cylinders $M(m) = M(F|K^m)$ ($1 \leq m \leq n-1$) of restrictions of F to the skeletons of K . There are expansions

$$\begin{aligned} K &\nearrow^2 M(1) \cup \left(\bigcup e_\beta^2 \right) \cup \left(\bigcup e_\gamma^3 \right) \cup \cdots \cup \left(\bigcup e_\nu^n \right) \\ &\nearrow^3 M(2) \cup \left(\bigcup e_\gamma^3 \right) \cup \cdots \cup \left(\bigcup e_\nu^n \right) \\ &\nearrow^4 \cdots \\ &\nearrow^n M(n-1) \cup \left(\bigcup e_\nu^n \right) = K_1, \end{aligned}$$

where e_μ^m ($m = 2, \dots, n$) are cells of K . Slide the attaching map of each n -cell e_ν^n in K_1 ($\nu \in \mathcal{J}_n$) along the walls of the mapping cylinder $M(n-1)$ in a level-preserving manner to obtain a homotopy n -deformation $K_1 \xrightarrow{n} K_2 = M(n-1) \cup (\bigcup \bar{e}_\nu^n)$, for \bar{e}_ν^n ($\nu \in \mathcal{J}_n$) as in Definition (1.2.2). Observe that the n -cells $\bar{e}_\nu^n \in K_2$ ($\nu \in \mathcal{J}_n$) are precisely the n -cells in K/T . Finally, perform cylindrical collapses

$$\begin{aligned} K_2 &= M(n-1) \cup \left(\bigcup \bar{e}_\nu^n \right) \\ &\searrow_1^n (K^{n-1}/T) \cup \left(\bigcup \bar{e}_\nu^n \right) = K/T. \quad \square \end{aligned}$$

We now turn to the notion of substitution, as motivated by Example (3.3.1). N.B. We use the additive notation (2.1.2) here even in the case $m = 3$, for convenience.

Suppose that $m \geq 3$. The *boundary equivalence relation* \doteq on words of order m is defined by $w_1 \doteq w_2$ if and only if $d_m(\langle w_1 \rangle) = d_m(\langle w_2 \rangle)$. Suppose that $\mathcal{W} = \langle \{w_\alpha^1\} | \cdots | \{w_\nu^n\} \rangle$ is a word system of order n , and let \hat{w} be a new m -word such that $\hat{w} \doteq (1, \lambda_0)$ for some $\lambda_0 \in \mathcal{J}_{m-1}$ where λ_0 is not mentioned in \hat{w} . We permit \hat{w} to mention new $(m-1)$ -indices $\eta_j \notin \mathcal{J}_{m-1}$ ($1 \leq j \leq J$), provided that boundaries $d_m(1, \eta_j) = \langle \overline{w}_{\eta_j}^{m-1} \rangle$ are also supplied.

We say that $\overline{\mathcal{W}} = \langle \{\overline{w}_\alpha^1\} | \cdots | \{\overline{w}_\nu^n\} \rangle$ arises from \mathcal{W} by *substituting* \hat{w} for λ_0 if $\overline{\mathcal{W}}$ satisfies the following conditions.

- (a) The index sets for $\overline{\mathcal{W}}$ are $\overline{\mathcal{J}}_{m-1} = \mathcal{J}_{m-1} - \{\lambda_0\} \cup \{\eta_1, \dots, \eta_J\}$ and $\overline{\mathcal{J}}_s = \mathcal{J}_s$ ($s \neq m-1$).
- (b) $\overline{w}_\sigma^s = w_\sigma^s$ for $s < m-1$, $s > m$ or ($s = m-1$ and $\sigma \neq \lambda_0$).
- (c) \overline{w}_μ^m ($\mu \in \mathcal{J}_m$) is formed by replacing each syllable $\varepsilon_q(g_q, \lambda_0)$ in w_μ^m by the word $\varepsilon_q g_q \cdot \hat{w}$.

Note that the $(m-1)$ -word $w_{\lambda_0}^{m-1}$ is omitted from $\overline{\mathcal{W}}$.

When $m = 2$, the definition of substitution is complicated by the fact that changing the 1-field can affect the coefficients g in higher-order syllables $\varepsilon(g, \rho)$.

Suppose that $\mathcal{W} = \langle \{w_\alpha^1\} | \cdots | \{w_\nu^n\} \rangle$ is a word system of order n , and let \hat{w} be a new 2-word which may mention new 1-indices $\eta_j \notin \mathcal{J}_1$. Then we say that $\overline{\mathcal{W}} = \langle \{\overline{w}_\alpha^1\} | \cdots | \{\overline{w}_\nu^n\} \rangle$ arises from \mathcal{W} by *substituting* \hat{w} for α_0 if $\overline{\mathcal{W}}$ satisfies the following conditions.

- (a) The index sets for $\overline{\mathcal{W}}$ are $\overline{\mathcal{J}}_1 = \mathcal{J}_1 - \{\alpha_0\} \cup \{\eta_1, \dots, \eta_J\}$ and $\overline{\mathcal{J}}_s = \mathcal{J}_s$ ($s > 1$).
- (b) $\overline{w}_\alpha^1 = w_\alpha^1$ for $\alpha \neq \alpha_0$.
- (c) \overline{w}_β^2 ($\beta \in \mathcal{J}_2$) is found by replacing each occurrence of α_0^ε in w_β^2 by the word $(\hat{w})^\varepsilon$ ($\varepsilon = \pm 1$).
- (d) \overline{w}_σ^s ($s \geq 3$, $\sigma \in \mathcal{J}_s$) is found by replacing each syllable $\varepsilon_q(g_q, \rho_q)$ in w_σ^s by the syllable $\varepsilon_q(\tilde{g}_q, \rho_q)$, where a representing word for \tilde{g}_q is obtained from a representing word for g_q by replacing each occurrence of α_0^ε by $(\hat{w})^\varepsilon$ ($\varepsilon = \pm 1$).

Observe that a substitution opportunity occurs whenever there is a word $w^l = \varepsilon_1(g_1, \kappa_1) + \cdots + \varepsilon_Q(g_Q, \kappa_Q)$ in \mathcal{W} with trivial boundary that mentions an $(l-1)$ -index κ exactly once. (When $l = 3$, the word would have the form $w^3 = (g_1, \kappa_1)^{\varepsilon_1} \cdots (g_Q, \kappa_Q)^{\varepsilon_Q}$. When $l = 2$, $w^2 = \kappa_1^{\varepsilon_1} \cdots \kappa_Q^{\varepsilon_Q}$.) To see this, let

$$\hat{w}^l = \begin{cases} -\varepsilon_q g_q^{-1} \cdot (\varepsilon_{q+1}(g_{q+1}, \kappa_{q+1}) + \cdots + \varepsilon_Q(g_Q, \kappa_Q) \\ \quad + \varepsilon_1(g_1, \kappa_1) + \cdots + \varepsilon_{q-1}(g_{q-1}, \kappa_{q-1})) & \text{when } l > 3, \\ g_q^{-1} \cdot ((g_{q+1}, \kappa_{q+1})^{\varepsilon_{q+1}} \cdots (g_Q, \kappa_Q)^{\varepsilon_Q} (g_1, \kappa_1)^{\varepsilon_1} \\ \quad \cdots (g_{q-1}, \kappa_{q-1})^{\varepsilon_{q-1}})^{-\varepsilon_q} & \text{when } l = 3, \\ (\kappa_q^{\varepsilon_{q+1}} \cdots \kappa_Q^{\varepsilon_Q} \kappa_1^{\varepsilon_1} \cdots \kappa_{q-1}^{\varepsilon_{q-1}})^{-\varepsilon_q} & \text{when } l = 2. \end{cases}$$

Then it is clear that $\hat{w}^l \doteq (1, \kappa_q)$ if $l \geq 3$ and that \hat{w}^l does not mention κ_q , so substitution is possible. We say that we are *substituting via* w^l . We also say

that \hat{w}^l arises from w^l by solving for $(1, \kappa_q)$ (or for κ_q , if $l = 2$).

Proposition (3.3.3) (Geometric dragging technique). *Let $m \geq 1$. Suppose that K is a geometric realization of a word system \mathcal{W} of order n such that each cell $e_\sigma^s \in K$ of dimension $s > m$ is a model cell for the corresponding word $w_\sigma^s \in \mathcal{W}$. Let $K^m \xrightarrow{e, m} K^m \cup \bar{e}_\mu^m - e_\mu^m$ be an elementary homotopy m -move of the m -skeleton of K with associated expansion cell \bar{e}_μ^{m+1} .*

Then K homotopy n -deforms to an n -complex \bar{K} that satisfies the following conditions.

- (a) $\bar{K}^m = K^m \cup \bar{e}_\mu^m - e_\mu^m$.
- (b) \bar{K} is a geometric realization of the word system $\bar{\mathcal{W}}$ that arises from \mathcal{W} by substitution via \tilde{w}_μ^{m+1} , where \tilde{w}_μ^{m+1} is a word for \bar{e}_μ^{m+1} that mentions the m -indices μ and $\bar{\mu}$ exactly once each.
- (c) The cells $\bar{e}_\sigma^s \in \bar{K}$ ($s > m$) are model cells for the corresponding words $\bar{w}_\sigma^s \in \bar{\mathcal{W}}$.

Proof. First, observe that Proposition (3.1.7) guarantees that a geometric word \tilde{w}_μ^{m+1} exists satisfying (b). Thus the word system $\bar{\mathcal{W}}$ is well defined. Note that a model map \tilde{f}_μ^{m+1} for the word \tilde{w}_μ^{m+1} exists, by Proposition (3.2.3).

Let \mathbf{I}^{m+1} denote $\text{bdy}(\mathbf{I}^m \times \mathbf{I})$. Adjust $\tilde{f}_\mu^{m+1}: \mathbf{I}^{m+1} \rightarrow K^m$ if necessary so that

- (1) $\tilde{f}_\mu^{m+1}|_{\mathbf{I}^m \times 0}$ is a characteristic map for e_μ^m , and
- (2) $\tilde{f}_\mu^{m+1}|_{\mathbf{I}^m \times 1}$ is a characteristic map for \bar{e}_μ^m .

The existence of such characteristic maps is guaranteed by the definition of a model map. Without loss of generality, \tilde{f}_μ^{m+1} may be taken as the attaching map for \bar{e}_μ^{m+1} .

We will proceed to slide the attaching maps of cells in K across the expansion cell \bar{e}_μ^{m+1} , one dimension at a time, noting the effect on words as we go.

Define the complex

$$K(m) = K \cup \{\bar{e}_\mu^m, \bar{e}_\mu^{m+1}\}.$$

Then the characteristic map $\tilde{\phi}_\mu^{m+1}: \mathbf{I}^{m+1} \rightarrow \text{cl}(\bar{e}_\mu^{m+1}) \subset K(m)$ is the homotopy $\tilde{\phi}_\mu^{m+1}: \phi_\mu^m \simeq \bar{\phi}_\mu^m$ of the homotopy m -deformation of K^m of the hypothesis. We may define a map $F_m: K^m \times \mathbf{I} \rightarrow K(m)$ by

$$F_m(x, t) = \begin{cases} x & \text{for } x \in K^m - e_\mu^m, \\ \tilde{\phi}_\mu^{m+1} \circ D_t((\phi_\mu^m)^{-1}(x), 0) & \text{for } x \in e_\mu^m \end{cases}$$

for a strong deformation retraction D_t from \mathbf{I}^{m+1} to $\text{cl}(\mathbf{I}^{m+1} - \{\mathbf{I}^m \times 0\})$. The map F_m is continuous by Lemma (3.1.1). As t varies from 0 to 1, the effect of F_m is to smoothly stretch the cell e_μ^m across the expansion cell \bar{e}_μ^{m+1} to the other side $\tilde{f}_\mu^{m+1}(\text{cl}(\mathbf{I}^{m+1} - \{\mathbf{I}^m \times 0\}))$, while holding the rest of K^m (including $\text{bdy}(e_\mu^m)$) fixed.

Now, slide the attaching maps of $(m+1)$ -cells $e_\xi^{m+1} \in K$ across the homotopy F_m to obtain a homotopy $(m+1)$ -move

$$K(m)^{m+1} = K(m)^m \cup \left(\bigcup_{\xi \in \mathcal{J}_{m+1}} e_\xi^{m+1} \right) \xrightarrow{m+1} K(m)^m \cup \left(\bigcup_{\xi \in \mathcal{J}_{m+1}} \bar{e}_\xi^{m+1} \right).$$

It is easy to check that the attaching maps \bar{f}_ξ^{m+1} for the new cells \bar{e}_ξ^{m+1} are model maps for the words \bar{w}_ξ^{m+1} , where \bar{w}_ξ^{m+1} arises from w_ξ^{m+1} by substituting via \tilde{w}_μ^{m+1} .

Inductively, let q satisfy $m+2 \leq q \leq n$. Suppose that a complex $K(q-2)$ and a homotopy $F_{q-2}: K^{q-2} \times \mathbf{I} \rightarrow K(q-2)$ have been defined such that there is a homotopy $(q-1)$ -deformation

$$K(q-2)^{q-2} \cup \left(\bigcup_{\alpha \in \mathcal{J}_{q-1}} e_\alpha^{q-1} \right) \xrightarrow{q-1} K(q-2)^{q-2} \cup \left(\bigcup_{\alpha \in \mathcal{J}_{q-1}} \bar{e}_\alpha^{q-1} \right).$$

Let \bar{e}_α^q be the expansion cell for the elementary homotopy $(q-1)$ -move from e_α^{q-1} to \bar{e}_α^{q-1} ($\alpha \in \mathcal{J}_{q-1}$), and define

$$K(q-1) = K(q-2) \cup \left(\bigcup_{\alpha \in \mathcal{J}_{q-1}} \{\bar{e}_\alpha^{q-1}, \bar{e}_\alpha^q\} \right).$$

Let F_{q-1} be the homotopy $F_{q-1}: K^{q-1} \times \mathbf{I} \rightarrow K(q-1)$ defined by

$$F_{q-1}(x, t) = \begin{cases} F_{q-2}(x, t), & \text{if } x \in K^{q-2}, \\ \tilde{\phi}_\alpha^q((\phi_\alpha^{q-1})^{-1}(x), t), & \text{if } x \in e_\alpha^{q-1}, \alpha \in \mathcal{J}_{q-1}. \end{cases}$$

(Note that no deformation retraction is interposed.) Then we may slide the attaching maps of q -cells e_β^q across F_{q-1} to obtain a homotopy q -deformation

$$K(q-1)^q = K(q-1)^{q-1} \cup \left(\bigcup_{\beta \in \mathcal{J}_q} e_\beta^q \right) \xrightarrow{q} K(q-1)^{q-1} \cup \left(\bigcup_{\beta \in \mathcal{J}_q} \bar{e}_\beta^q \right).$$

The attaching maps $\bar{f}_\beta^q: \mathbf{I}^q \rightarrow K(q-1)^{q-1}$ of new q -cells \bar{e}_β^q are defined by $\bar{f}_\beta^q(y) = F_{q-1}(f_\beta^q(y), 1)$ ($y \in \mathbf{I}^q$), and each \bar{f}_β^q is clearly a model map for \bar{w}_β^q , the word that arises from w_β^q by substituting via \tilde{w}_μ^{m+1} .

Define $K(n) = K(n-1)^{n-1} \cup (\bigcup_{\nu \in \mathcal{J}_n} \bar{e}_\nu^n)$, and let $\bar{\mathcal{W}}$ arise from \mathcal{W} by substituting via \tilde{w}_μ^{m+1} . Then there is a homotopy n -deformation

$$\begin{aligned} K & \nearrow_{e, m+1} K(m) \nearrow^{m+2} K(m+1) \nearrow^{m+3} \cdots \nearrow^n K(n-1) \xrightarrow{n} K(n) \\ & \searrow \bar{K} = (K^m \cup \bar{e}_\mu^m - e_\mu^m) \cup \left(\bigcup_{\xi \in \mathcal{J}_{m+1}} \bar{e}_\xi^{m+1} \right) \cup \cdots \cup \left(\bigcup_{\nu \in \mathcal{J}_n} \bar{e}_\nu^n \right) \end{aligned}$$

where \bar{K} is a geometric realization of $\bar{\mathcal{W}}$. \square

We now present two applications of geometric dragging.

Proposition (3.3.4) (Existence of model complexes). *If a geometric realization K exists for a word system \mathcal{W} of order n , then K homotopy n -deforms to a model complex $K(\mathcal{W})$ for \mathcal{W} .*

Proof. Internally collapse a maximal tree (3.3.2) to obtain a complex \tilde{K} with only one vertex such that $K \nearrow^n \tilde{K}$. By Proposition (3.2.3) and geometric

dragging (3.3.3) we may deform \tilde{K} to a complex $K(n)$ whose n -cells are model cells. Inductively by the same reasoning, there is a homotopy deformation $K(m+1) \xrightarrow{n} K(m)$ ($2 \leq m \leq n-1$) to a complex $K(m)$ whose cells \bar{e}_σ^s of dimension $s \geq m$ are model cells. Therefore, $K \xrightarrow{n} K(2)$, where $K(2)$ is clearly a model complex for \mathcal{W} . \square

Proposition (3.3.5). *If K and \bar{K} are geometric realizations of the same word system \mathcal{W} of order n , then $K \xrightarrow{n} \bar{K}$.*

Proof. By Proposition (3.3.4), we may assume that K and \bar{K} are model complexes without loss of generality.

Perform an elementary 1-expansion $K \nearrow K \cup \{\bar{e}^0, \bar{e}^1\}$, where \bar{e}^1 is a 1-cell joining the unique 0-cells $e^0 \in K$ and $\bar{e}^0 \in \bar{K}$. Slide the (trivial) attaching maps of 1-cells $e_\alpha^1 \in K$ along \bar{e}^1 to obtain a homotopy 1-move

$$(K)^1 \cup \{\bar{e}^0, \bar{e}^1\} \xrightarrow{1} (\bar{K})^1 \cup \{e^0, \bar{e}^1\}.$$

Then $K \cup \{\bar{e}^0, \bar{e}^1\} \xrightarrow{n} L \xrightarrow{e,1} K(1)$ where L is obtained by geometric dragging and $K(1) = L - \{e^0, \bar{e}^1\}$. Observe that $K(1)$ is a model complex for \mathcal{W} with $K(1)^1 = (\bar{K})^1$.

Continue this process in each higher dimension. Since the attaching maps f_μ^m for $e_\mu^m \in K(1)$ and \bar{f}_μ^m for $\bar{e}_\mu^m \in \bar{K}$ are associated with the same word $w_\mu^m \in \mathcal{W}$, we see $f_\mu^m \simeq \bar{f}_\mu^m$. Thus, there is a homotopy m -deformation

$$K(m-1)^m \xrightarrow{m} K(m-1)^m \cup \left(\bigcup_\mu \bar{e}_\mu^m \right) - \left(\bigcup_\mu e_\mu^m \right).$$

Then $K(m-1) \xrightarrow{n} K(m)$ by geometric dragging (3.3.3), where $K(m)$ satisfies $K(m)^m = \bar{K}^m$.

It follows by induction that K homotopy n -deforms to the complex $\bar{K} = K(n)$. \square

4. ALGEBRAIC FORM OF THE MAIN THEOREM

4.1 Algebraic Equivalence Theorem. We will prove the following result, which establishes most of Theorem (6.1.1).

Theorem (4.1.1) (Algebraic Equivalence Theorem). *Let \mathcal{W} and \mathcal{W}' be word systems of order n . Then the following are equivalent:*

- (1) \mathcal{W} is P_n -equivalent to \mathcal{W}' .
- (2) \mathcal{W} is fully P_n -equivalent to \mathcal{W}' .
- (3) \mathcal{W} is D_{n+1} -equivalent to \mathcal{W}' .

In this section we will show the implication (3) \Rightarrow (1), which is the central idea behind the result. The remainder of the proof involves technical arguments that are summarized in the next section.

Theorem (4.1.2). *If \mathscr{W} is D_{n+1} -equivalent to \mathscr{W}' then \mathscr{W} is P_n -equivalent to \mathscr{W}' .*

Proof. Let $\mathscr{W} = \mathscr{W}_0, \mathscr{W}_1, \dots, \mathscr{W}_R = \mathscr{W}'$ be a D_{n+1} -sequence from \mathscr{W} to \mathscr{W}' . We will use the notation $w_\mu^m(\mathscr{W}_r)$ when necessary to specify which word system \mathscr{W}_r contains a given word w_μ^m .

Since \mathscr{W}_0 is a word system of order n , every $(n+1)$ -word that occurs in a word system \mathscr{W}_r ($1 \leq r \leq R-1$) must arise from an $(n+1)$ -expansion prior to the move $[\mathscr{W}_r, \mathscr{W}_{r+1}]$ in the D_{n+1} -sequence. Hence, that $(n+1)$ -word must have been introduced together with a new n -word for its expansion face. For bookkeeping purposes, we will choose nonzero integers that satisfy the following conditions to be the indices of n -words and $(n+1)$ -words of \mathscr{W}_r .

Indexing conditions on \mathscr{W}_r . (a) Positive integer indices, indicating the order of appearance in the D_{n+1} -sequence $\mathscr{W} = \mathscr{W}_0, \mathscr{W}_1, \dots, \mathscr{W}_r$, are assigned to any $(n+1)$ -words w_{+i}^{n+1} that are *present in \mathscr{W}_r* , and the same indices are assigned to the corresponding expansion face n -words $w_{+i}^n \in \mathscr{W}_r$. (b) Negative integer indices are assigned to all n -words of \mathscr{W}_r that are not included in case (a).

If \mathscr{W}_r satisfies the indexing conditions, define the word system

$$\mathscr{V}_r = \mathscr{W}_r - \bigcup_{+i>0} \{w_{+i}^n, w_{+i}^{n+1}\}.$$

\mathscr{V}_r is the result of collapsing all $(n+1)$ -words $w_{+i}^{n+1} \in \mathscr{W}_r$ in reverse order of appearance through their expansion face n -words $w_{+i}^n \in \mathscr{W}_r$. Choose negative integer indices for all n -words in $\mathscr{W} = \mathscr{W}_0$.

Suppose that \mathscr{W}_r satisfies the indexing conditions, and assume that a P_n -sequence exists from $\mathscr{V}_0 = \mathscr{W}$ to the subsystem \mathscr{V}_r of \mathscr{W}_r . We shall find indexing on \mathscr{W}_{r+1} satisfying the indexing conditions such that \mathscr{V}_r is P_n -equivalent to \mathscr{V}_{r+1} . We consider each case of the D_{n+1} -move $[\mathscr{W}_r, \mathscr{W}_{r+1}]$.

Case 1. Inversion, $(w_\mu^m)^{-1}$. Choose the same indexing for \mathscr{W}_{r+1} as for \mathscr{W}_r in order to preserve the indexing conditions. If $m \leq n$, then \mathscr{V}_{r+1} results from \mathscr{V}_r by the same inversion $(w_\mu^m)^{-1}$. If $m = n+1$ then $\mathscr{V}_r = \mathscr{V}_{r+1}$.

Case 2. Elementary m -deformation, $m \leq n$. Select indexing for \mathscr{W}_{r+1} that is consistent with the indexing for \mathscr{W}_r . In the case of an n -expansion, assign a negative integer index to the expansion n -word. Then \mathscr{V}_{r+1} is obtained from \mathscr{V}_r by the same elementary m -deformation $[\mathscr{W}_r, \mathscr{W}_{r+1}]$.

Case 3. Elementary $(n+1)$ -expansion. Assign the index $+r$ to both new words w_{+r}^n and w_{+r}^{n+1} . For other words, choose the same indexing for \mathscr{W}_{r+1} as for \mathscr{W}_r . The indexing conditions are satisfied and we have $\mathscr{V}_{r+1} = \mathscr{V}_r$.

Case 4. Elementary $(n+1)$ -collapse, with w_{+k}^{n+1} and w_β^n removed. If the collapse face n -index β is positive, then we must have $\beta = +k$. This is because w_β^{n+1} is present in \mathscr{W}_r , and thus w_β^n cannot be a free face of w_{+k}^{n+1} unless $\beta = +k$. Choose the same indexing in \mathscr{W}_{r+1} as in \mathscr{W}_r and observe that $\mathscr{V}_{r+1} = \mathscr{V}_r$.

Suppose that the collapse face n -index β is negative, $\beta = -j$. Then the collapse face index $-j \in \mathcal{I}_n$ differs from the expansion face index $+k \in \mathcal{I}_n$ for $w_{+k}^{n+1}(\mathscr{W}_r)$. Assign indices in \mathscr{W}_{r+1} consistent with the indexing in \mathscr{W}_r , except reindex $w_{+k}^n(\mathscr{W}_r)$ as $w_{-j}^n(\mathscr{W}_{r+1})$. Then $w_{-j}^n(\mathscr{W}_{r+1})$ is the expansion face and $w_{-j}^n(\mathscr{W}_r)$ is the collapse face for w_{+k}^{n+1} , and we seek a P_n -sequence from \mathscr{V}_r to $\mathscr{V}_{r+1} = \mathscr{V}_r \cup \{w_{-j}^n(\mathscr{W}_{r+1})\} - \{w_{-j}^n(\mathscr{W}_r)\}$.

Solve $w_{+k}^{n+1} \in \mathcal{W}_r$ for $(1, +k)$ in the sense of Subsection 3.3 to obtain a word $\hat{w}_{+k}^{n+1} \doteq (1, +k)$. Then \hat{w}_{+k}^{n+1} mentions the index $-j$ exactly once, and any positive index $+i$ occurring in \hat{w}_{+k}^{n+1} satisfies $+i < +k$ by the indexing criteria. Let $+i_0$ be the highest such index $+i$. Then we may solve $w_{+i_0}^{n+1}$ for $(1, +i_0)$ to obtain $\hat{w}_{+i_0}^{n+1} \doteq (1, +i_0)$, where again the only positive integer indices $+h$ occurring in $\hat{w}_{+i_0}^{n+1}$ satisfy $+h < +i_0$. Now, substitute $\hat{w}_{+i_0}^{n+1}$ for $(1, +i_0)$ in \hat{w}_{+k}^{n+1} , yielding a word $W_1 \doteq (1, +k)$ such that all positive integers $+a$ mentioned in W_1 satisfy $+a < +i_0$. Furthermore, W_1 still mentions $-j$ exactly once, since that index cannot be mentioned in $\hat{w}_{+i_0}^{n+1}$ as w_{-j}^n is a free face of w_{+k}^{n+1} .

Continue this process, defining $(n+1)$ -words $W_2 \doteq W_3 \doteq \dots$ with decreasing highest positive integer n -index. The process must stop in a word W_l that contains no positive integer n -indices. We have $W_l \doteq (1, +k)$, and W_l mentions the index $-j$ associated with the word system \mathcal{W}_r exactly once.

Now, write W_l in the form

$$W_l = \varepsilon_1(g_1, -p_1) + \dots + \varepsilon_{q-1}(g_{q-1}, -p_{q-1}) + \varepsilon_q(g_q, -j) \\ + \varepsilon_{q+1}(g_{q+1}, -p_{q+1}) + \dots + \varepsilon_Q(g_Q, -p_Q).$$

When $n \leq 3$, we write

$$W_l = \begin{cases} (g_1, -p_1)^{\varepsilon_1} \dots (g_{q-1}, -p_{q-1})^{\varepsilon_{q-1}} (g_q, -j)^{\varepsilon_q} (g_{q+1}, -p_{q+1})^{\varepsilon_{q+1}} \\ \dots (g_Q, -p_Q)^{\varepsilon_Q} & \text{when } n = 3, \\ (-p_1)^{\varepsilon_1} \dots (-p_{q-1})^{\varepsilon_{q-1}} (-j)^{\varepsilon_q} (-p_{q+1})^{\varepsilon_{q+1}} \dots (-p_Q)^{\varepsilon_Q} & \text{when } n = 2. \end{cases}$$

Then we may use the form of W_l as a guide to find a sequence of P_n -moves from \mathcal{V}_r to \mathcal{V}_{r+1} , as follows.

1. Conjugate $w_{-j}^n \in \mathcal{V}_r$ by the group element g_q , yielding $g_q \cdot w_{-j}^n$ (conjugation is omitted when $n = 2$).
2. Invert $g_q \cdot w_{-j}^n$ if necessary to obtain $\varepsilon_q g_q \cdot w_{-j}^n$ (or $g_q \cdot (w_{-j}^n)^{\varepsilon_q}$ when $n = 3$, or $(w_{-j}^n)^{\varepsilon_q}$ when $n = 2$).
3. Conjugate and invert $w_{-p_{q+1}}^n$ to obtain $\varepsilon_{q+1} g_{q+1} \cdot w_{-p_{q+1}}^n$.
4. Perform an n -product equivalence yielding $\varepsilon_q g_q \cdot w_{-j}^n + \varepsilon_{q+1} g_{q+1} \cdot w_{-p_{q+1}}^n$ (or an analogous form if $n = 2$ or 3).
5. Conjugate and invert $\varepsilon_{q+1} g_{q+1} \cdot w_{-p_{q+1}}^n$ back to $w_{-p_{q+1}}^n$.
6. Conjugate and invert $w_{-p_{q+2}}^n$ to obtain $\varepsilon_{q+2} g_{q+2} \cdot w_{-p_{q+2}}^n$.
7. Perform an n -product equivalence yielding

$$\varepsilon_q g_q \cdot w_{-j}^n + \varepsilon_{q+1} g_{q+1} \cdot w_{-p_{q+1}}^n + \varepsilon_{q+2} g_{q+2} \cdot w_{-p_{q+2}}^n.$$

8. Continue with similar conjugations, inversions and n -product equivalences.

Then \mathcal{V}_r is P_n -equivalent to the word system $\overline{\mathcal{V}}_r = \mathcal{V}_r \cup \{\overline{w}^n\} - \{w_{-j}^n\}$, where the n -word \overline{w}^n has the form

$$\overline{w}^n = \varepsilon_1 g_1 \cdot w_{-p_1}^n + \dots + \varepsilon_{q-1} g_{q-1} \cdot w_{-p_{q-1}}^n + \varepsilon_q g_q \cdot (w_{-j}^n) \\ + \varepsilon_{q+1} g_{q+1} \cdot w_{-p_{q+1}}^n + \dots + \varepsilon_Q g_Q \cdot w_{-p_Q}^n$$

(or an analogous form if $n = 2$ or 3). Clearly, $\langle \overline{w}^n \rangle = d_{n+1}(\langle W_l \rangle)$. Since $W_l \doteq w_{+k}^n$ and $\langle w_{+k}^n \rangle = d_{n+1}(\langle (1, +k) \rangle)$, we conclude that \overline{w}^n represents the

same element in $M_n(\mathcal{J}_{n-1})$ as the word $w_{+k}^n(\mathcal{W}_r) = w_{-j}^n(\mathcal{W}_{r+1})$. Thus, a word-representative equivalence transforms $\overline{\mathcal{V}}_r$ into \mathcal{V}_{r+1} . Hence there exists a P_n -sequence from \mathcal{V}_r to \mathcal{V}_{r+1} .

We conclude by induction that there is a P_n -sequence from $\mathcal{V}_0 = \mathcal{W}$ to $\mathcal{V}_R = \mathcal{W}'$. \square

5

5.1 Remainder of the proof of the Algebraic Equivalence Theorem. Since P_n -equivalence implies full P_n -equivalence, Theorem (4.1.1) will follow if we can show that fully P_n -equivalent word systems are D_{n+1} -equivalent. We will show this using an algebraic counterpart of Proposition (3.3.3), called algebraic dragging (5.1.2). The easiest way to establish algebraic dragging is by converting to a geometric problem and applying (3.3.3). Unfortunately, this cannot be done in general for higher-order word systems, as Example (2.2.3) shows. We will take this geometric approach in low dimensions in order to avoid some unpleasant noncommutative algebra.

Lemma (5.1.1). *Let $\mathcal{W} \nearrow_{e,m} \mathcal{W}' = \mathcal{W} \cup \{w_\lambda^{m-1}, w_\mu^m\}$ be an elementary m -expansion of word systems. If $K = K(\mathcal{W})$ is a model complex for \mathcal{W} , then $K \nearrow_{e,m} K'$ where $K' = K(\mathcal{W}')$ is a model complex for \mathcal{W}' .*

Proof. It suffices to show that model cells e_λ^{m-1} for w_λ^{m-1} and e_μ^m for w_μ^m exist. This is straightforward when $m = 2$.

Suppose $m \geq 3$. Solve the word w_μ^m for $(1, \lambda)$ in the sense of Subsection 3.3, yielding $\hat{w}_\mu^m \doteq (1, \lambda)$. Construct spaces $X \subset N \subset S^{m-1}$ and a map $\tilde{F}: N \rightarrow K^{m-1}$ such that \hat{w}_μ^m is associated with \tilde{F} , $\hat{w}_\mu^m \sim \tilde{F}$, as in the proof of Proposition (3.2.3). The neighborhood N is a ball, so $\partial \tilde{F}: (\text{bdy}(N), *) \rightarrow (K^{m-2}, *)$ represents an element $[\partial \tilde{F}] \in \pi_{m-2}(K^{m-2}, K^{m-3})$. Trivially, $w_\lambda^{m-1} \sim \partial \tilde{F}$. In fact, $\partial \tilde{F}$ actually represents an element in the absolute homotopy group $\pi_{m-2}(K^{m-2})$, so by the long exact sequence in homotopy, $\beta([\partial \tilde{F}]) \in \pi_{m-3}(K^{m-3})$ is the trivial element. Therefore, we may apply (3.2.3) to obtain a homotopy $H: \partial \tilde{F} \simeq f_\lambda^{m-1}$ to a model map f_λ^{m-1} for w_λ^{m-1} . Let e_λ^{m-1} be a new $(m-1)$ -cell attached to K^{m-2} by f_λ^{m-1} .

Let $\overline{\phi}_\lambda^{m-1}$ be the characteristic map for e_λ^{m-1} with orientation reversed. Then we may paste together the map \tilde{F} , the homotopy H and a copy of $\overline{\phi}_\lambda^{m-1}$ to obtain a model map $F_1: S^{m-1} \rightarrow K^{m-1}$ for the word $w_1 = -(1, \lambda) + \hat{w}_\mu^m$ (making trivial choices of basepoint and connecting path for e_λ^{m-1}). Modify F_1 on a small neighborhood of the basepoint of S^{m-1} , yielding a map $F_2: S^{m-1} \rightarrow K^{m-1}$ that is a model map for the word $w_2 = -(g, \lambda) + g \cdot \hat{w}_\mu^m$, where g is the coefficient of λ in the expansion word $w_\mu^m \in \mathcal{W}$. Clearly, w_2 is a cyclic permutation of either w_μ^m or $-w_\mu^m$. Therefore, either F_2 or \overline{F}_2 is a model map $f_\mu^m: S^{m-1} \rightarrow K^{m-1}$ for the word w_μ^m , by Proposition (3.2.2). Attach a new cell e_μ^m to K^{m-1} via f_μ^m . Then we have constructed model cells e_λ^{m-1} and e_μ^m . \square

Proposition (5.1.2) (Algebraic dragging technique). *Let \mathscr{W} be a word system of order n , and let $w_\mu^m \in \mathscr{W}$. Suppose that \bar{w}_μ^m and \tilde{w}_μ^{m+1} are words representing elements in $M_m(\mathscr{W})$ and $M_{m+1}(\mathscr{W})$ such that*

- (a) \bar{w}_μ^m has a new m -index $\bar{\mu} \notin \mathcal{I}_m(\mathscr{W})$ and satisfies $d(\langle \bar{w}_\mu^m \rangle) = 1$, and
- (b) \tilde{w}_μ^{m+1} has a new $(m+1)$ -index $\tilde{\mu} \notin \mathcal{I}_{m+1}(\mathscr{W})$, satisfies $d(\langle \tilde{w}_\mu^{m+1} \rangle) = 1$ and mentions the m -indices μ and $\bar{\mu}$ exactly once each.

Let $\overline{\mathscr{W}}$ arise from \mathscr{W} by substituting via \tilde{w}_μ^{m+1} , solving for $(1, \mu)$ (or solving for μ in the case $n = 2$).

Then $\mathscr{W} \xrightarrow[n+1]{n+1} \overline{\mathscr{W}}$.

We first establish the following special case.

Lemma (5.1.3). *Proposition (5.1.2) holds if \mathscr{W} has a geometric realization. Furthermore, in that case we may rearrange the D_{n+1} -sequence so that all expansions occur first, i.e., the deformation has the form $\mathscr{W} \xrightarrow[n+1]{n+1} \mathscr{W}' \xrightarrow[n+1]{n+1} \overline{\mathscr{W}}$.*

Proof of lemma. By Proposition (3.3.4) a model complex $K = K(\mathscr{W})$ exists for \mathscr{W} . Let \bar{e}_μ^m and \tilde{e}_μ^{m+1} be model cells reflecting the expansion $\mathscr{W} \xrightarrow[n+1]{n+1} \mathscr{W}'$ $\nearrow_{e, m+1}$ $\mathscr{W} \cup \{\bar{w}_\mu^m, \tilde{w}_\mu^{m+1}\}$ as in Lemma (5.1.1). Examining the proof, we see that there is a homotopy m -move $K^m \xrightarrow[e, m]{} K^m \cup \bar{e}_\mu^m - e_\mu^m$ and that the elementary $(m+1)$ -expansion associated with this homotopy move involves the cells \bar{e}_μ^m and \tilde{e}_μ^{m+1} . Apply Proposition (3.3.3) to obtain a homotopy n -deformation $K \xrightarrow[n]{n} \bar{K}$ where $\bar{K} = K(\overline{\mathscr{W}})$ is a model complex for $\overline{\mathscr{W}}$. Then there is an associated formal $(n+1)$ -deformation $K \xrightarrow[n+1]{n+1} \bar{K}$.

That deformation may be rearranged so that all expansions come first

$$K \xrightarrow[n+1]{n+1} K' \xrightarrow[n+1]{n+1} \bar{K}$$

by a standard argument. An examination of the proof of Proposition (3.3.3) shows that K' is a model complex for a word system \mathscr{W}' . Now apply the idea of Corollary (3.1.8) to obtain $\mathscr{W} \xrightarrow[n+1]{n+1} \mathscr{W}' \xrightarrow[n+1]{n+1} \overline{\mathscr{W}}$. \square

Proof of Proposition (5.1.2). Let w_ρ^r denote a typical word in \mathscr{W} . After reindexing, $\overline{\mathscr{W}}$ consists of:

- w_ρ^r ($r < m$);
- w_ζ^m ($\zeta \neq \mu$) and \bar{w}_μ^m ; and
- \bar{w}_ρ^r ($r > m$), where \bar{w}_ρ^r arises from w_ρ^r by substituting for μ via \tilde{w}_μ^{m+1} .

It is enough to define expansion words \tilde{w}_ρ^{r+1} ($\rho \in \mathcal{I}_r$, $m+1 \leq r \leq n$) satisfying the conditions

- (i) \tilde{w}_ρ^{r+1} mentions the indices ρ and $\bar{\rho}$ exactly once each, and
- (ii) $d_{r+1}(\langle \tilde{w}_\rho^{r+1} \rangle)$ is trivial,

for then we may deform \mathscr{W} to $\overline{\mathscr{W}}$ as in (5.1.4).

$$(5.1.4) \quad \begin{aligned} \mathscr{W} \nearrow \mathscr{W} \cup \{\overline{w}_\mu^m, \tilde{w}_\mu^{m+1}\} \cup \left(\bigcup_{\xi \in \mathcal{J}_{m+1}} \{\overline{w}_\xi^{m+1}, \tilde{w}_\xi^{m+2}\} \right) \\ \cup \dots \cup \left(\bigcup_{\nu \in \mathcal{J}_n} \{\overline{w}_\nu^n, \tilde{w}_\nu^{n+1}\} \right) \searrow \overline{\mathscr{W}}. \end{aligned}$$

We argue by induction on r . As the initial case we claim that expansion words \tilde{w}_ρ^{r+1} exist for $r = \max(m+1, 4)$. When $m < 4$, consider the 4-section $\mathscr{W}^4 = \{w_\pi^p \in \mathscr{W} \mid p \leq 4\}$. \mathscr{W}^4 has a geometric realization by Proposition (2.2.4), so we may apply the lemma to obtain a 5-expansion $\mathscr{W}^4 \xrightarrow[m+1]{5} \mathscr{W}'$, where

$$\mathscr{W}' = \mathscr{W}^4 \cup \{\overline{w}_\mu^m, \tilde{w}_\mu^{m+1}\} \cup \left(\bigcup_{\xi \in \mathcal{J}_{m+1}} \{\overline{w}_\xi^{m+1}, \tilde{w}_\xi^{m+2}\} \right) \cup \dots \cup \left(\bigcup_{\delta \in \mathcal{J}_4} \{\overline{w}_\delta^4, \tilde{w}_\delta^5\} \right).$$

An examination of the proof of Proposition (3.3.3) and the lemma shows

$$\tilde{w}_\delta^5 = (1, \delta) - (1, \bar{\delta}) - w_\delta^5(\widetilde{\mathcal{J}}_3).$$

Here we write $w_\lambda^{l+1}(\widetilde{\mathcal{J}}_{l-1})$ to denote the result of replacing each $(l-1)$ -index κ in w_λ^l by the l -index $\tilde{\kappa}$. Then \tilde{w}_δ^5 satisfies condition (i), and the boundary condition (ii) holds by the lemma.

When $m \geq 4$, we may write \tilde{w}_μ^{m+1} in the form $\tilde{w}_\mu^{m+1} = \varepsilon(g, \mu) + \varepsilon g \cdot \hat{w}_\mu^{m+1}$ where \hat{w}_μ^{m+1} arises from \tilde{w}_μ^{m+1} by solving for $(1, \mu)$. If w_ξ^{m+1} is any $(m+1)$ -word, it can be written in the form $w_\xi^{m+1} = A_0 + \varepsilon_1(g_1, \mu) + A_1 + \dots + \varepsilon_Q(g_Q, \mu) + A_Q$. Define

$$\tilde{w}_\xi^{m+2} = (1, \xi) - (1, \bar{\xi}) - \varepsilon g^{-1} \cdot (\varepsilon_1(g_1, \tilde{\mu}) + \dots + \varepsilon_Q(g_Q, \tilde{\mu})).$$

This word satisfies (i), and (ii) follows by the same reasoning as in the inductive step below.

Assume that words \tilde{w}_ρ^{r+1} ($\rho \in \mathcal{J}_r$) have been defined, where

$$\max(m+1, 4) \leq r < n.$$

Then define the words ($\sigma \in \mathcal{J}_{r+1}$)

$$\tilde{w}_\sigma^{r+2} = (1, \sigma) - (1, \bar{\sigma}) - w_\sigma^{r+2}(\widetilde{\mathcal{J}}_r)$$

representing an element in the module $M_{r+2}(\mathcal{J}_{r+1} \cup \overline{\mathcal{J}}_{r+1} \cup \widetilde{\mathcal{J}}_r)$. Using commutativity and the definition of substitution, we see that $d_{r+2}(\langle w_\sigma^{r+2}(\widetilde{\mathcal{J}}_r) \rangle)$ is just the word needed to convert w_σ^{r+1} to $\overline{w}_\sigma^{r+1}$ in the commutative module $M_{r+1}(\mathcal{J}_r \cup \overline{\mathcal{J}}_r \cup \widetilde{\mathcal{J}}_{r-1})$, so $d_{r+2}(\langle \tilde{w}_\sigma^{r+2} \rangle)$ is trivial. Thus \tilde{w}_σ^{r+2} satisfies (i) and (ii). This completes the induction. \square

Remark (5.1.5). The D_{n+1} -sequence $\mathscr{W} \bigwedge_{m+1}^{n+1} \overline{\mathscr{W}}$ that results from algebraic dragging (Proposition (5.1.2)) may be assumed transient.

To see this, we examine the proof and note that $(n+1)$ -words can arise in the proof of Proposition (5.1.2) only in the following instances.

1. If \mathcal{W} has a geometric realization, then $(n+1)$ -words arise in the proof of Lemma (5.1.3). Those words correspond to expansion $(n+1)$ -cells associated with a homotopy n -deformation $K \xrightarrow[n]{m} \bar{K}$ of model complexes K, \bar{K} of dimension n . Lemma (1.2.3) implies that the associated formal $(n+1)$ -deformation $K \xrightarrow{n+1} \bar{K}$ is transient, so a transient D_{n+1} -sequence exists in that case.

2. In the proof of Proposition (5.1.2) for the general case, the deformation (5.1.4) can clearly be rearranged into a transient deformation. \square

Proposition (5.1.6). *Suppose that $\mathcal{W} = \mathcal{W}_0, \dots, \mathcal{W}_Q = \bar{\mathcal{W}}$ is a full P_n -sequence of word systems of order n . Then there is a D_{n+1} -sequence from \mathcal{W} to $\bar{\mathcal{W}}$.*

Proof. We argue by induction on q ($0 \leq q \leq Q$). Suppose that a D_{n+1} -sequence exists from \mathcal{W}_0 to \mathcal{W}_q . If the full P_n -move $[\mathcal{W}_q, \mathcal{W}_{q+1}]$ is an inversion, elementary m -expansion or elementary m -collapse ($m \leq n$), then we can immediately extend the D_{n+1} -sequence to \mathcal{W}_{q+1} .

Now consider the other full P_n -moves. Suppose that $\bar{w}_\mu^m = \alpha \cdot w_\mu^m$ is an m -conjugation. Then $\mathcal{W}_q \xrightarrow[e, m+1]{} \mathcal{W}_q \cup \{\bar{w}_\mu^m, \tilde{w}_\mu^{m+1}\}$, where $\tilde{w}_\mu^{m+1} = (1, \bar{\mu}) - (\alpha, \mu)$ (or $\tilde{w}_\mu^3 = \bar{\mu}\alpha\mu^{-1}\alpha^{-1}$ when $m = 2$). Perform algebraic dragging (5.1.2) to obtain a D_{n+1} -sequence of word systems $\mathcal{W}_q \cup \{\bar{w}_\mu^m, \tilde{w}_\mu^{m+1}\} \xrightarrow[n+1]{} \bar{\mathcal{W}}_q$. It is clear that $\bar{\mathcal{W}}_q$ satisfies $\bar{\mathcal{W}}_q = \mathcal{W}_{q+1} \cup \{w_\mu^m, \tilde{w}_\mu^{m+1}\}$, so we may collapse $\bar{\mathcal{W}}_q \xrightarrow[e, m]{} \mathcal{W}_{q+1}$.

A similar argument applies for the remaining full P_n -moves. If $w_\mu^m \sim \bar{w}_\mu^m$ is a word-representative equivalence, then choose $\tilde{w}_\mu^{m+1} = (1, \bar{\mu}) - (1, \mu)$ (or $\bar{\mu}\mu^{-1}$ when $m = 2$). For m -product equivalence, $\bar{\mu} = \mu\tau$, use $\tilde{w}_\mu^{m+1} = (1, \bar{\mu}) - (1, \tau) - (1, \mu)$ (or $\tilde{w}_\mu^3 = \bar{\mu}\tau^{-1}\mu^{-1}$ when $m = 2$). Then we have obtained a D_{n+1} -sequence from \mathcal{W}_q to \mathcal{W}_{q+1} for every full P_n -move $[\mathcal{W}_q, \mathcal{W}_{q+1}]$. \square

Remark (5.1.7). Observe that the D_{n+1} -sequence of Proposition (5.1.6) may be assumed transient.

This holds because $(n+1)$ -words w^{n+1} that occur in the proof of that proposition satisfy one of the following conditions.

1. $w^{n+1} = \tilde{w}_\mu^{m+1}$ where $m = n$. In this case, no algebraic dragging takes place, and the sequence $\mathcal{W}_q \xrightarrow[e, m+1]{} \mathcal{W}_q \cup \{\bar{w}_\mu^m, \tilde{w}_\mu^{m+1}\} \xrightarrow[e, m]{} \mathcal{W}_{q+1}$ is transient.
2. w^{n+1} arises during algebraic dragging. Remark (5.1.5) shows that $\mathcal{W}_q \xrightarrow[n+1]{} \mathcal{W}_{q+1}$ may be assumed transient. \square

Proof of Theorem (4.1.1). Theorem (4.1.2) establishes (D_{n+1} -equivalence $\Rightarrow P_n$ -equivalence), and the implication (P_n -equivalence \Rightarrow full P_n -equivalence) is trivial. The implication (full P_n -equivalence $\Rightarrow D_{n+1}$ -equivalence) follows from Proposition (5.1.6). \square

6. MAIN THEOREM

6.1 The Formal Equivalence Theorem. We are now ready to prove the following.

Theorem (6.1.1) (Formal Equivalence Theorem). *Suppose that K and K' are CW n -complexes with homotopy readings \mathscr{W} and \mathscr{W}' , respectively.*

Then the following are equivalent:

- (a) \mathscr{W} and \mathscr{W}' are P_n -equivalent.
- (b) \mathscr{W} and \mathscr{W}' are fully P_n -equivalent.
- (c) \mathscr{W} and \mathscr{W}' are D_{n+1} -equivalent.
- (d) K formally $(n+1)$ -deforms to K' .

Proof. Theorem (4.1.1) shows (a) \Leftrightarrow (b) \Leftrightarrow (c). Furthermore, suppose that \mathscr{W} is D_{n+1} -equivalent to \mathscr{W}' . Since K is a geometric realization of \mathscr{W} , there is a model complex $K(\mathscr{W})$ such that $K \xrightarrow{n} K(\mathscr{W})$ by Proposition (3.3.4).

It follows that $K \xrightarrow{n+1} K(\mathscr{W})$ by Lemma (1.2.3). Then we can emulate every D_{n+1} -move from \mathscr{W} to \mathscr{W}' by a corresponding move between model complexes, using Lemma (5.1.1) for expansions, the definition of a model complex for collapses and orientation reversal of characteristic maps for inversion. Then $K(\mathscr{W}) \xrightarrow{n+1} K(\mathscr{W}')$ where $K(\mathscr{W}')$ is a model complex for \mathscr{W}' . Finally, $K(\mathscr{W}') \xrightarrow{n+1} K'$ using Proposition (3.3.5), so there is a formal $(n+1)$ -deformation from K to K' . This shows (c) \Rightarrow (d).

It suffices to show (d) \Rightarrow (b). Let $K = K_0, K_1, \dots, K_Q = K'$ be a formal deformation of n -complexes K and K' , and suppose that \mathscr{W} is a homotopy reading of K . Then \mathscr{W} is fully P_n -equivalent to a word system \mathscr{W}_0 that is a homotopy reading of K consisting of geometric words, by Propositions (3.1.7) and (2.3.8). We now show that \mathscr{W}_0 is fully P_n -equivalent to a homotopy reading \mathscr{W}_Q of K' .

Let $T \subset K$ be the maximal tree that was chosen for the homotopy reading \mathscr{W}_0 . We may insist that T satisfy the following condition, for $q = 0$:

- (*) $T_q \subset K_q$ does not contain any 1-cells that will be removed as a collapse face for any 2-cell e_β^2 present in K_q .

(N.B. Some 1-cells in T_q may well be collapse faces for future 2-cells that arise later in $K_q \xrightarrow{n+1} K'$.) Note that this condition can be verified by a connectivity argument involving the complex

$$(K_q)^1 \cup \{e^2 \in K_q \mid e^2 \text{ will be collapsed during } K_q \hookrightarrow K'\}.$$

Suppose that for some $q \geq 0$, we already have a homotopy reading \mathscr{W}_q of K_q with chosen maximal tree $T_q \subset (K_q)^1$ such that \mathscr{W}_q consists of geometric words and T_q satisfies (*). By Corollary (3.1.8) and the definition of geometric words, it is clear that an elementary expansion $K_q \xrightarrow[e, m]{} K_{q+1}$ or collapse $K_q \xrightarrow[e, m]{} K_{q+1}$ of complexes induces an elementary expansion $\mathscr{W}_q \xrightarrow[e, m]{} \mathscr{W}_{q+1}$ or collapse $\mathscr{W}_q \xrightarrow[e, m]{} \mathscr{W}_{q+1}$ of word systems, where \mathscr{W}_{q+1} is a homotopy reading of K_{q+1} consisting of geometric words. Also, the tree T_{q+1} chosen for the homotopy

reading \mathcal{W}_{q+1} must satisfy $(*)$, except in one case. That case occurs if a 2-expansion or 3-expansion introduces a new 2-cell e^2 that has a collapse face e^1 in T_{q+1} .

In that exceptional case, we apply the connectivity argument again to find a maximal tree \bar{T}_{q+1} that satisfies $(*)$. Apply Lemma (2.3.7) to obtain a word system $\bar{\mathcal{W}}_{q+1}$ that is a homotopy reading of K_{q+1} using the maximal tree \bar{T}_{q+1} , such that \mathcal{W}_{q+1} is P_1 -equivalent to $\bar{\mathcal{W}}_{q+1}$. Lemma (2.3.7) (and free reduction in the case of 2-words) further asserts that $\bar{\mathcal{W}}_{q+1}$ consists of geometric words, because \mathcal{W}_{q+1} does. Thus, \mathcal{W}_q is fully P_m -equivalent to $\bar{\mathcal{W}}_{q+1}$ where $\bar{\mathcal{W}}_{q+1}$ and \bar{T}_{q+1} satisfy the induction hypothesis.

We conclude that \mathcal{W} is fully P_n -equivalent to a homotopy reading \mathcal{W}_Q of K' . Since \mathcal{W}_Q and \mathcal{W}' are two homotopy readings for the same complex, they are fully P_n -equivalent by Proposition (2.3.8). This establishes $(d) \Rightarrow (b)$ and completes the proof of the Formal Equivalence Theorem. \square

7. APPLICATIONS

Our proof of Theorem (6.1.1) does not rely on transiency. By this, we mean that an accumulation of top-dimensional words is permitted in the proof of Theorem (4.1.2) (D_{n+1} -equivalence \Rightarrow P_n -equivalence). However, an analysis of other parts of the argument establishes the following transiency properties for all dimensions.

Proposition (7.1.1). *Suppose that \mathcal{W} and $\bar{\mathcal{W}}$ are word systems of order n . If there is a D_{n+1} -sequence from \mathcal{W} to $\bar{\mathcal{W}}$ then there is a transient D_{n+1} -sequence from \mathcal{W} to $\bar{\mathcal{W}}$.*

Proof. There exists a P_n -sequence from \mathcal{W} to $\bar{\mathcal{W}}$ by Theorem (6.1.1). Proposition (5.1.6) shows how to convert that P_n -sequence to a D_{n+1} -sequence, which may be assumed transient by Remark (5.1.7). \square

Proposition (7.1.2). *Suppose that K and \bar{K} are CW n -complexes. If a formal $(n+1)$ -deformation $K \overset{n+1}{\frown} \bar{K}$ exists, then there is a transient formal $(n+1)$ -deformation from K to \bar{K} .*

Proof. Let \mathcal{W} and $\bar{\mathcal{W}}$ be homotopy readings of K and \bar{K} , respectively. By Proposition (3.3.4), there exist homotopy n -deformations $K \overset{n}{\frown} K(\mathcal{W})$ and $\bar{K} \overset{n}{\frown} K(\bar{\mathcal{W}})$ to model complexes $K(\mathcal{W})$ and $K(\bar{\mathcal{W}})$ of \mathcal{W} and $\bar{\mathcal{W}}$. Thus, there are transient formal $(n+1)$ -deformations $K \overset{n+1}{\frown} K(\mathcal{W})$ and $\bar{K} \overset{n+1}{\frown} K(\bar{\mathcal{W}})$ by Lemma (1.2.3).

Applying Theorem (6.1.1) and Proposition (7.1.1), we conclude that there is a transient D_{n+1} -sequence from \mathcal{W} to $\bar{\mathcal{W}}$. We can perform the corresponding sequence of expansions and collapses starting with $K(\mathcal{W})$ to obtain a transient formal $(n+1)$ -deformation $K(\mathcal{W}) \overset{n+1}{\frown} K'$, where K' is a model complex for $\bar{\mathcal{W}}$. Proposition (3.3.5) and Lemma (1.2.3) imply that there is a transient formal $(n+1)$ -deformation $K' \overset{n+1}{\frown} K(\bar{\mathcal{W}})$. Thus, we have a transient formal

$(n + 1)$ -deformation

$$K \frown^{n+1} K(\mathscr{W}) \frown^{n+1} K' \frown^{n+1} K(\overline{\mathscr{W}}) \frown^{n+1} \overline{K}. \quad \square$$

We close with a comment on nonrealizable word systems, such as Example (2.2.3).

Proposition (7.1.3). *In a P_n -equivalence class of word systems, either all of the word systems have geometric realizations or none of them do.*

Proof. Suppose \mathscr{W} is P_n -equivalent to \mathscr{W}' , and let K be a geometric realization of \mathscr{W} . Then there is a model complex $K(\mathscr{W})$ for \mathscr{W} , by Proposition (3.3.4). Also, \mathscr{W} is D_{n+1} -equivalent to \mathscr{W}' by Theorem (4.1.1). Mimicking these D_{n+1} -moves as in the proof of Theorem (6.1.1), we obtain a deformation $K(\mathscr{W}) \frown^{n+1} K(\mathscr{W}')$ to a model complex for \mathscr{W}' . \square

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